LECTURE NOTES

ON

DYNAMICS OF MACHINERY

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III B.Tech I Semester

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UNIT 1
FRICTION

Introduction

It has been established since long, that the surfaces of the bodies are never perfectly smooth. When, even a very smooth surface is viewed under a microscope, it is found to have roughness and irregularities, which may not be detected by an ordinary touch. If a block of one substance is placed over the level surface of the same or of different material, a certain degree of interlocking of the minutely projecting particles takes place. This does not involve any force, so long as the block does not move or tends to move. But whenever one block moves or tends to move tangentially with respect to the surface, on which it rests, the interlocking property of the projecting particles opposes the motion. This opposing force, which acts in the opposite direction of the movement of the upper block, is called the ***force of friction*** or simply ***friction***. It thus follows, that at every joint in a machine, force of friction arises due to the relative motion between two parts and hence some energy is wasted in overcoming the friction. Though the friction is considered undesirable, yet it plays an important role both in nature and in engineering *e.g.* waling on a road, motion of locomotive on rails, transmission of power by belts, gears etc. The friction between the wheels and the road is essential for the car to moveforward.

Types of Friction

In general, the friction is of the following two types :

1. *Static friction.* It is the friction, experienced by a body, when atrest.

2. *Dynamicfriction.*Itisthefriction,experiencedby abody,wheninmotion.Thedynamicfrictionisalsocalled ***kinetic friction*** and is less than the static friction. It is of the following three types :

(a) *Sliding friction.* It is the friction, experienced by a body, when it ***slides*** over anotherbody.

(b) Rolling friction. It is the friction, experienced between the surfaces which has ***balls*** or***rollers*** interposed between them.

(c) *Pivot friction.* It is the friction, experienced by a body, due to the ***motion of rotation*** as in case of foot stepbearings.

The friction may further be classified as :

1. Friction between unlubricated surfaces,and

2. Friction between lubricatedsurfaces.

Friction BetweenUnlubricated Surfaces

The friction experienced between two dry and unlubricated surfaces in contact is known as ***dry*** or ***solid friction***. It is due to the surface roughness. The dry or solid friction includes the sliding friction and rolling friction as discussedabove.

Friction Between Lubricated Surfaces

When lubricant (*i.e.* oil or grease) is applied between two surfaces in contact, then the friction may be classified into the following two types depending upon the thickness of layer of alubricant.

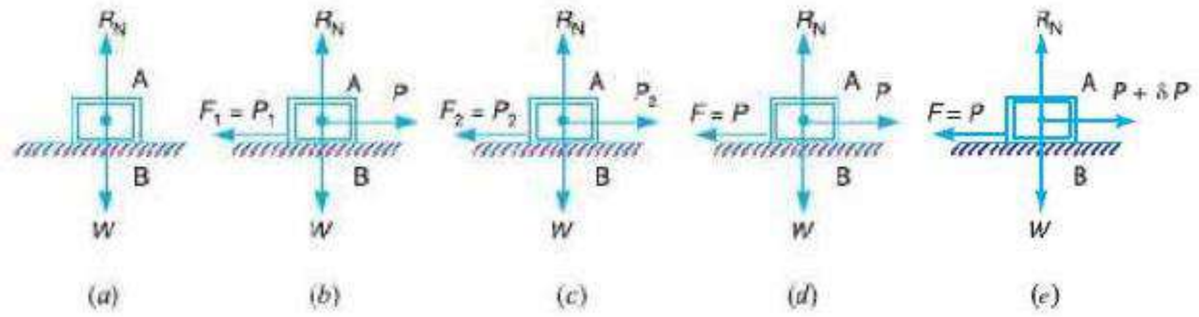
1. *Boundary friction* (***or greasy friction or non-viscous friction***). It is the friction, experienced between the rubbing surfaces, when the surfaces have a very thin layer of lubricant. The thickness of this very thin layer is of the molecular dimension. In this type of friction, a thin layer of lubricant forms a bond between the two rubbing surfaces. The lubricant is absorbed on the surfaces and forms a thin film. This thin film of the lubricant results in less friction between them. The boundary friction follows the laws of solidfriction.

2. *Fluid friction* (*or film friction or viscous friction*). It is the friction, experienced between the rubbing surfaces, when the surfaces have a thick layer of the lubrhicant. In this case, the actual surfaces do not come in contact and thus do not rub against each other. It is thus obvious that fluid friction is not due to the surfaces in contact but it is due to the ***viscosity*** and ***oiliness*** of the lubricant. **Note :**The ***viscosity*** is a measure of the resistance offered to the sliding one layer of the lubricant over an adjacent layer. The absolute viscosity of a lubricant may be defined as the force required to cause a plate of unit area to slide with unit velocity relative to a parallel plate, when the two plates are separated by a layer of lubricant of unitthickness.

The ***oiliness*** property of a lubricant may be clearly understood by considering two lubricants of equal viscosities and atequaltemperatures.Whentheselubricantsaresmearedontwodifferentsurfaces,itisfoundthattheforceoffrictionwithone lubricantisdifferentthanthatoftheother.Thisdifferenceisduetothepropertyofthelubricantknownasoiliness.Thelubricant which gives lower force of friction is said to have greateroiliness.

Limiting Friction

Consider that a body *A* of weight *W* is lying on a rough horizontal body *B*as shown in Fig. In this position, the body *A*is in equilibrium under the action of its own weight *W* , and the normal reaction R_N (equal to *W*) of *B* on *A* . Now if a small horizontal force P_1 is applied to the body *A* acting through its centre of gravity as shown in Fig., it does not move because of the frictional force which prevents the motion. This shows that the applied force P_1 is exactly balanced by the force of friction F_1 acting in the opposite direction If we now increase the applied force to P_2as shown in Fig. 10.1 (*c*), it is still found to be in equilibrium. This means that the force of friction has also increased to a value $F_2 = P_2$. Thus every time the effort is increased the force of friction also increases, so as to become exactly equal to the applied force. There is, however, a limit beyond which the force of friction cannot increase as shown in Fig. 10.1 (*d*). After this, any increase in the applied effort will not lead to any further increase in the force of friction, as shown in Fig. 10.1 (*e*), thus the body *A* begins to move in the direction of the applied force. This maximum value of frictional force, which comes into play, when a body just begins to slide over the surface of the other body, is known as ***limiting force of friction*** or simply ***limiting friction***. It may be noted that when the applied force is less than the limiting friction, the body remains at rest, and the friction into play is called **static friction** which may have any value between zero and limitingfriction.



Laws of Static Friction

Following are the laws of static friction :

1. The force of friction always acts in a direction, opposite to that in which the body tends tomove.

2. The magnitude of the force of friction is exactly equal to the force, which tends the body tomove.

3. The magnitude of the limiting friction (F) bears a constant ratio to the normal reaction (R_N) between the two surfaces.Mathematically

$$F/R_N = \text{constant}$$

4. The force of friction is independent of the area of contact, between the twosurfaces.

5. The force of friction depends upon the roughness of thesurfaces.

Laws of Kinetic or Dynamic Friction

Following are the laws of kinetic or dynamic friction :

1. The force of friction always acts in a direction, opposite to that in which the body ismoving.

2. The magnitude of the kinetic friction bears a constant ratio to the normal reaction between the two surfaces. But this ratio is slightly less than that in case of limitingfriction.

3. For moderate speeds, the force of friction remains constant. But it decreases slightlywith the increase of speed.

Laws of Solid Friction

Following are the laws of solid friction :

1. The force of friction is directly proportional to the normal load between thesurfaces.

2. The force of friction is independent of the area of the contact surface for a given normalload.

3. The force of friction depends upon the material of which the contact surfaces aremade.

4. The force of friction is independent of the velocity of sliding of one body relative to the otherbody.

Laws of Fluid Friction

Following are the laws of fluid friction :

1. The force of friction is almost independent of theload.

2. The force of friction reduces with the increase of the temperature of thelubricant.

3. The force of friction is independent of the substances of the bearingsurfaces.

4. The force of friction is different for differentlubricants.

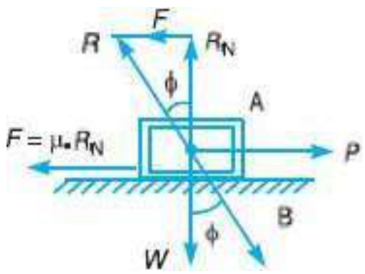
Coefficient of Friction

It is defined as the ratio of the limiting friction (*F*) to the normal reaction (R_N) between the two bodies. It is generally denoted by ∝. Mathematically, coefficient offriction,

$$\propto = F/R_N$$

Limiting Angle of Friction

Consider that a body *A* of weight (*W*) is resting on a horizontal plane *B*, as shown in Fig. 10.2. If a horizontal force *P* is applied to the body, no relative motion will take place until the applied force *P* is equal to the force of friction *F*, acting opposite to the direction of motion. The magnitude of this force of friction is $F = \propto.W = \propto.R_N$, where R_N is the normal reaction. In the limiting case, when the motion just begins, the body will be in equilibrium under the action of the following three forces:



1. Weight of the body (*W*),

2. Applied horizontal force (*P*),and

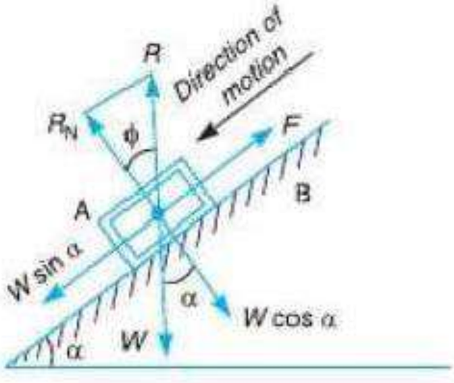
3. Reaction (*R*) between the body *A* and the plane*B*.

The reaction *R* must, therefore, be equal and opposite to the resultant of *W* and *P* and will be inclined at an angle ⎞ to the normal reaction R_N. This angle ⎞ is known as the ***limiting angle of friction***. It may be defined as the angle which the resultant reaction *R* makes with the normal reactionR_N.

From Fig. 10.2, tan ⎞ = $F/R_N = \propto R_N / R_N = \propto$

Angle of Repose

Consider that a body *A* of weight (*W*) is resting on an inclined plane *B*, as shown in Fig. 10.3. If the angle of inclination 〈 of the plane to the horizontal is such that the body begins to move down the plane, then the angle 〈 is called the ***angle of repose***. A little consideration will show that the body will begin to move down the plane when theangleofinclinationofthe planeisequaltotheangleoffriction(*i.e.*〈=⎞).Thismaybeprovedasfollows:



The weight of the body (*W*) can be resolved into the following two components :

1. *W* sin 〈, parallel to the plane *B*. This component tends to slide the body down theplane.

2. *W* cos〈, perpendicular to the plane *B*. This component is balanced by the normal reaction (R_N) of the bodyA and the planeB.

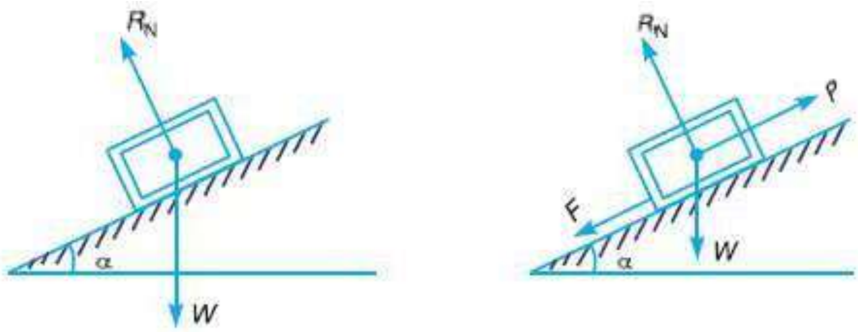
The body will only begin to move down the plane, when

W sin 〈 = *F* = ∝.R_N = ∝.*W* cos〈

tan〈 = ∝ = tan 〉 or 〈 = 〉

Friction of a Body Lying on a Rough Inclined Plane

Consider that a body of weight (*W*) is lying on a plane inclined at an angle 〈 with the horizontal, as shown in Fig. (a) and(*b*).



(*a*) Angle of inclination less than angle of friction.

(*b*) Angle of inclination more than angle of friction.

little consideration will show that if the inclination of the plane, with the horizontal, is less than the angle of friction, the body will be in equilibrium as shown in Fig. 10.6 (*a*). If,in this condi- tion, the body is required to be moved upwards and downwards, a corresponding force is required for the same. But, if the inclination of the plane is more than the angle of friction, the body will move down and an upward force (*P*) will be required to resist the body from moving down the plane as shown in Fig. (*b*).

Let us now analyse the various forces which act on a body when it slides either up or down an inclined plane.

1. ***Considering the motion of the body up theplane***

Let W = Weight of thebody,

α=Angleofinclinationoftheplanetothehorizontal,

ϕ=Limitingangleoffrictionforthecontactsurfaces,

P = Effort applied in a given direction in order to cause the body to slide with uniform velocity parallel to the plane, considering friction,

P_0 = Effort required to move the body up the plane neglecting friction,

θ = Angle which the line of action of P makes with the weight of the body W ,

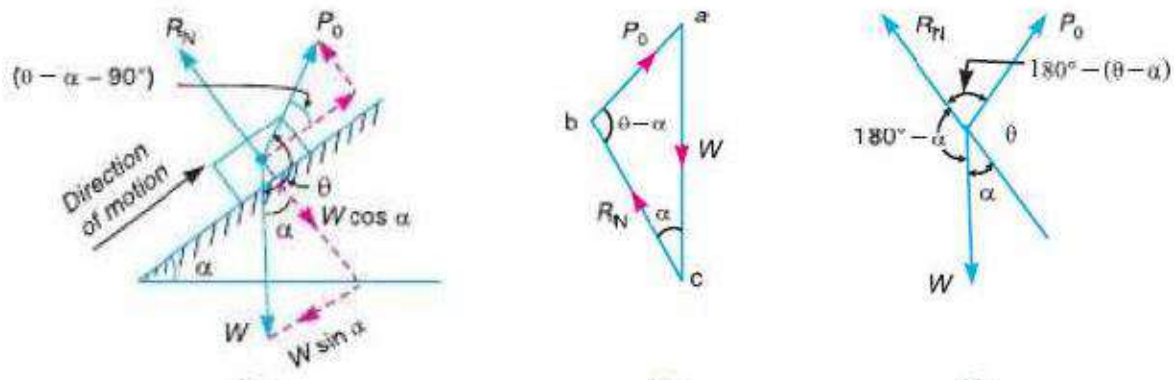
μ = Coefficient of friction between the surfaces of the plane and the body,

R_N = Normal reaction, and

R = Resultant reaction

When the friction is neglected, the body is in equilibrium under the action of the three forces, *i.e.* P_0, W and R_N, as shown in Fig. 10.7 (*a*). The triangle of forces is shown in Fig. 10.7 (*b*). Now applying sine rule for these three concurrent forces,

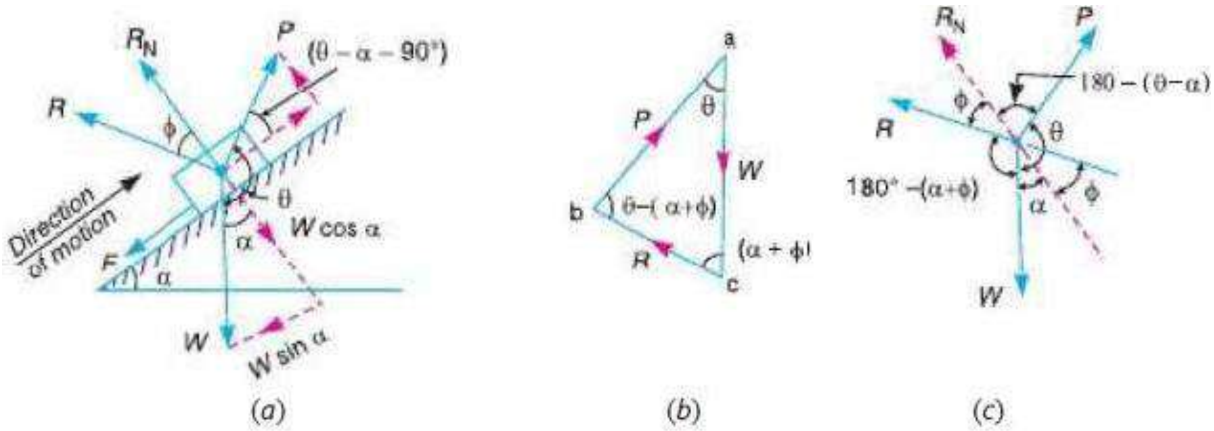
$$\frac{P_0}{\sin\alpha} = \frac{W}{\sin(\theta - \alpha)} \quad \text{or} \quad P_0 = \frac{W \sin\alpha}{\sin(\theta - \alpha)}$$



(*a*) (*b*) (*c*)

When friction is taken into account, a frictional force $F = \mu.R_N$ acts in the direction opposite to the motion of the body, as shown in Fig. 10.8 (*a*). The resultant reaction$R$ between the plane and the body is inclined at an angle ϕwiththenormalreactionR_N.Thetriangleofforcesisshowninfig.(*b*).Nowapplyingsinerule,

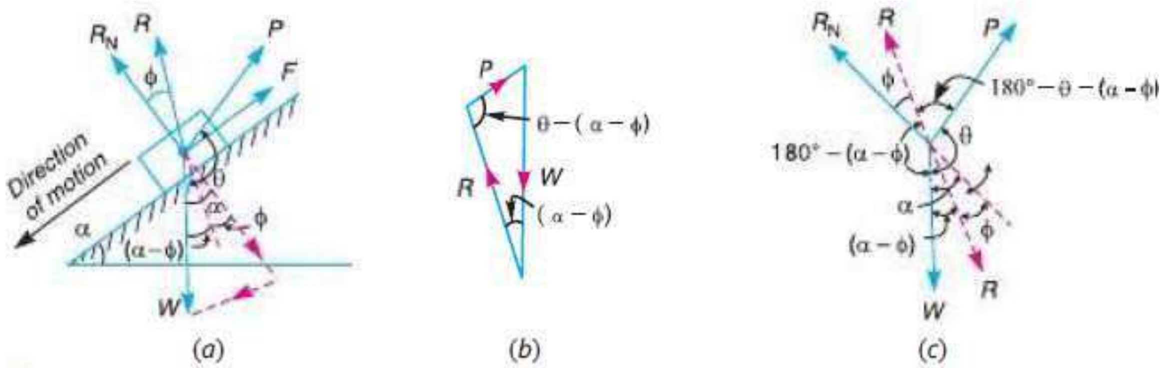
$$\frac{P}{\sin(\alpha + \phi)} = \frac{W}{\sin[\theta - (\alpha + \phi)]}$$



2. Considering the motion of the body down theplane

Neglecting friction, the effort required for the motion down the plane will be same as for the motion up the plane, *i.e.*

$$P_0 = \frac{W \sin ⟨}{\sin (⟨ ↕ ⟨)}$$



When the friction is taken into account, the force of friction $F = \propto . R_N$ will act up the plane and the resultant reaction R will make an angle ⟩ with R_N towards its right as shown in Fig. (*a*). The triangle of forces is shown in Fig. (*b*). Now from sine rule,

$$\frac{P}{\sin (⟨ ↕ ⟩)} = \frac{W}{\sin[⟨ ↕ (⟨ ↕ ⟩)]}$$

$$P = \frac{W \sin (⟨ ↕ ⟩)}{\sin[⟨ ↕ (⟨ ↕ ⟩)]}$$

Efficiency of Inclined Plane

The ratio of the effort required neglecting friction (*i.e.* P_0) to the effort required considering friction (*i.e.* P) is known as efficiency of the inclined plane. Mathematically, efficiency of the inclinedplane,

$$\eta = P_0 / P$$

Let us consider the following two cases :

1. For the motion of the body up theplane

$$\text{Efficiency,} \quad \eta = \frac{P_0}{P} = \frac{W \sin \alpha}{\sin(\theta - \alpha)} \times \frac{\sin[\theta - (\alpha + \phi)]}{W \sin(\alpha + \phi)}$$

$$= \frac{\sin \alpha}{\sin \theta \cos \alpha - \cos \theta \sin \alpha} \times \frac{\sin \theta \cos(\alpha + \phi) - \cos \theta \sin(\alpha + \phi)}{\sin(\alpha + \phi)}$$

Multiplying the numerator and denominator by $\sin(\alpha + \phi) \sin \theta$, we get

$$\eta = \frac{\cot(\alpha + \phi) - \cot \theta}{\cot \alpha - \cot \theta}$$

2. For the motion of the body down theplane

Since the value of P will be less than P_0, for the motion of the body down the plane, therefore in this case,

$$\eta = \frac{P}{P_0} = \frac{W \sin(\alpha - \phi)}{\sin[\theta - (\alpha - \phi)]} \times \frac{\sin(\theta - \alpha)}{W \sin \alpha}$$

$$= \frac{\sin(\alpha - \phi)}{\sin \theta \cos(\alpha - \phi) - \cos \theta \sin(\alpha - \phi)} \times \frac{\sin \theta \cos \alpha - \cos \theta \sin \alpha}{\sin \alpha}$$

Multiplying the numerator and denominator by $\sin(\alpha - \phi) \sin \theta$, we get

$$\eta = \frac{\cot \alpha - \cot \theta}{\cot(\alpha - \phi) - \cot \theta}$$

Screw Friction

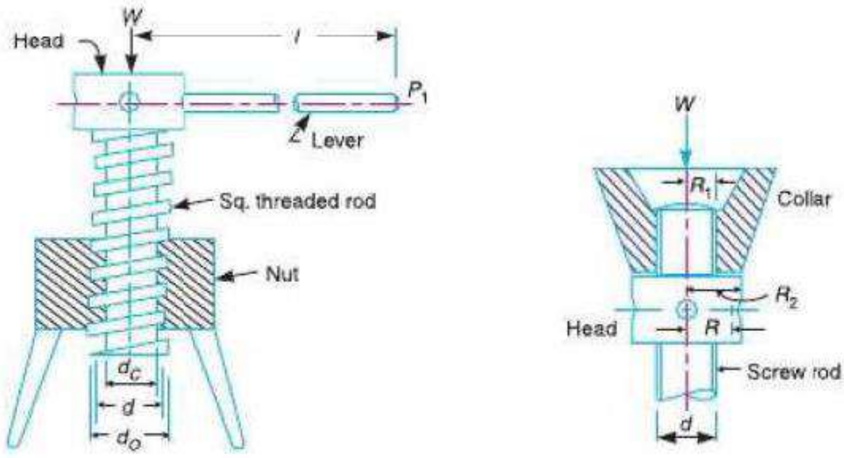
The screws, bolts, studs, nuts etc. are widely used in various machines and structures for temporary fastenings. These fastenings have screw threads, which are made by cutting a continuous helical groove on a cylindrical surface. If the threads are cut on the outer surface of a solid rod, these are known as ***external threads***. But if the threads are cut on the internal surface of a hollow rod, these are known as ***internal threads***. The screw threads are mainly of two types *i.e.* V-threads and square threads. The V-threads are stronger and offer more frictional resistance to motion than square threads. Moreover, the V-threads have an advantage of preventing the nut fromslackening.Ingeneral,theV-threadsareusedforthepurposeoftighteningpiecestogether *e.g.*boltsandnuts etc. But the square threads are used in screw jacks, vice screws etc. The following terms are important for the study of screw:

1. ***Helix.*** It is the curve traced by a particle, while describing a circular path at a uniformspeed and advancing in the axial direction at a uniform rate. In other words, it is the curve traced bya particle while moving along a screwthread.
2. ***Pitch***. It is the distance from a point of a screw to a corresponding point on the next thread, measured parallel to the axis of thescrew.
3. ***Lead***. It is the distance, a screw thread advances axially in oneturn.
4. ***Depth of thread***. It is the distance between the top and bottom surfaces of a thread (also known as**crest** and**root** of a thread).
5. ***Single-threadedscrew***.Iftheleadofascrewisequaltoitspitch,itisknownassinglethreadedscrew.
6. ***Multi-threaded screw.*** If more than one thread is cut in one lead distance of a screw, it is known as multi-threaded screw ***e.g.*** in a double threaded screw, two threads are cut in one lead length. In such cases, all the threads run independently along the length of the rod.Mathematically,
   Lead = Pitch × Number of threads
7. ***Helix angle.*** It is the slope or inclination of the thread with the horizontal.Mathematically,

$$\tan \langle = \frac{\text{Lead of screw}}{\text{Circumference of screw}}$$

Screw Jack

The screw jack is a device, for lifting heavy loads, by applying a comparatively smaller effort at its handle. The principle, on which a screw jack works is similar to that of an inclined plane.

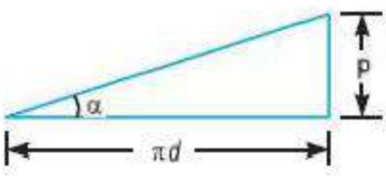


(a) Screw jack. (b) Thrust collar.

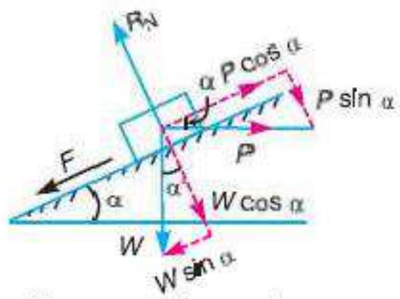
Fig. (*a*) shows a common form of a screw jack, which consists of a square threaded rod (also called screw rod or simply screw) which fits into the inner threads of the nut. The load, to be raised or lowered, is placed on the head of the square threaded rod which is rotated by the application of an effort at the end of the lever for lifting or lowering theload.

Torque Required to Lift the Load by a Screw Jack

If one complete turn of a screw thread by imagined to be unwound, from the body of the screw and developed, it will form an inclined plane as shown in Fig.(*a*).



(*a*) Development of a screw.



(*b*) Forces acting on the screw.

Let p = Pitch of the screw,

d = Mean diameter of the screw,

α = Helix angle,

P = Effort applied at the circumference of the screw to lift the load,

W = Load to be lifted, and

μ = Coefficient of friction, between the screw and nut = tan ϕ,where ϕ is the friction angle.

From the geometry of the Fig. 10.12 (*a*), we find that

$$\tan\alpha = p/\pi d$$

Since the principle on which a screw jack works is similar to that of an inclined plane, there- fore the force applied on the lever of a screw jack may be considered to be horizontal as shown in Fig. (*b*).

Since the load is being lifted, therefore the force of friction ($F = \mu.R_N$) will act downwards. All the forces acting on the screw are shown in Fig.(*b*).

Resolving the forces along the plane,

$$P\cos\alpha = W\sin\alpha + F = W\sin\alpha + \mu.R_N$$

and resolving the forces perpendicular to the plane,

$$R_N = P\sin\alpha + W\cos\alpha$$

Substituting this value of R_N in equation (*i*),

$$P\cos\alpha = W\sin\alpha + \mu(P\sin\alpha + W\cos\alpha)$$
$$= W\sin\alpha + \mu P\sin\alpha + \mu W\cos\alpha$$

4 $$P = W \cdot \frac{\sin \langle + \propto \cos \langle}{\cos \langle \updownarrow \propto \sin \langle}$$

Substituting the value of $\propto = \tan \rangle$ in the above equation, we get

$$P = W \cdot \frac{\sin \langle + \tan \rangle \cos \langle}{\cos \langle \updownarrow \tan \rangle \sin \langle}$$

Multiplying the numerator and denominator by $\cos \rangle$,

$$P = W \cdot \frac{\sin \langle \cos \rangle + \sin \rangle \cos \langle}{\cos \langle \cos \rangle \updownarrow \sin \langle \sin \rangle} = W \frac{\sin (\langle + \rangle)}{\cos (\langle + \rangle)}$$

$$= W \tan (\langle + \rangle)$$

4 Torque required to overcome friction between the screw and nut,

$$T_1 = P \cdot \frac{d}{2} = W \tan (\langle + \rangle) \frac{d}{2}$$

When the axial load is taken up by a thrust collar or a flat surface, as shown in Fig. 10.11 (*b*), so that the load does not rotate with the screw, then the torque required to overcome friction at the collar,

$$T_2 = \propto_1 W \left(\frac{R_1 + R_2}{2} \right) = \propto_1 . W . R$$

where R_1 and R_2 = Outside and inside radii of the collar,

R = Mean radius of the collar, and

$\propto_1$ = Coefficient of friction for the collar.

4 Total torque required to overcome friction (*i.e.* to rotate the screw),

$$T = T_1 + T_2 = P \times \frac{d}{2} + \propto_1 . W . R$$

If an effort P_1 is applied at the end of a lever of arm length l, then the total torque required to overcome friction must be equal to the torque applied at the end of the lever, *i.e.*

$$T = P \cdot \frac{d}{2} = P_1 . l$$

Torque Required to Lower the Load by a Screw Jack

We have discussed in Art. 10.18, that the principle on which the screw jack works is similar to that of an inclined plane. If one complete turn of a screw thread be imagined to be unwound from the body of the screw and developed, it will form an inclined plane as shown in Fig.(*a*).

Let p = Pitch of the screw,

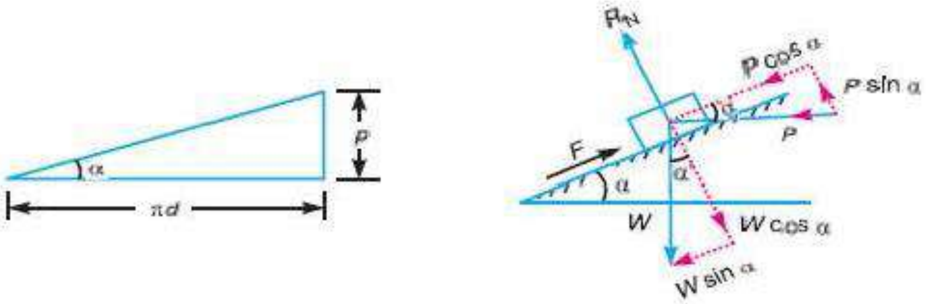
d = Mean diameter of the screw,

$\langle$ = Helix angle,

P = Effort applied at the circumference of the screw to lower the load,

W = Weight to be lowered, and

$\propto$ = Coefficient of friction between the screw and nut = $\tan \rangle$, where $\rangle$ is the friction angle.



(a) (b)

From the geometry of the figure, we find that

$$\tan\alpha = p/\pi d$$

Since the load is being lowered, therefore the force of friction ($F = \mu.R_N$) will act upwards. All the forces acting on the screw are shown in Fig.(*b*).

Resolving the forces along the plane,

$$P\cos\alpha = F - W\sin\alpha = \mu.R_N - W\sin\alpha$$

and resolving the forces perpendicular to the plane,

$$R_N = W\cos\alpha - P\sin\alpha$$

Substituting this value of R_N in equation ***(i)***,

$$P\cos\alpha = \mu\,(W\cos\alpha - P\sin\alpha) - W\sin\alpha$$
$$= \mu.W\cos\alpha - \mu.P\sin\alpha - W\sin\alpha$$

$$P = W\cdot\frac{(\mu\cos\alpha - \sin\alpha)}{(\cos\alpha + \mu\sin\alpha)}$$

Substituting the value of $\mu = \tan\phi$ in the above equation, we get

$$P = W\cdot\frac{(\tan\phi\cos\alpha - \sin\alpha)}{(\cos\alpha + \tan\phi\sin\alpha)}$$

Multiplying the numerator and denominator by $\cos\phi$,

$$P = W\cdot\frac{(\sin\phi\cos\alpha - \sin\alpha\cos\phi)}{(\cos\alpha\cos\phi + \sin\phi\sin\alpha)} = W\frac{\sin(\phi-\alpha)}{\cos(\phi-\alpha)}$$
$$= W\tan(\phi-\alpha)$$

Torque required to overcome friction between the screw and nut,

$$T = P\cdot\frac{d}{2} = W\tan(\phi-\alpha)\frac{d}{2}$$

Over Hauling and Self Locking Screws

We have seen in Art. 10.20 that the effort required at the circumference of the screw to lower the load is

$$P = W \tan(\phi - \alpha)$$

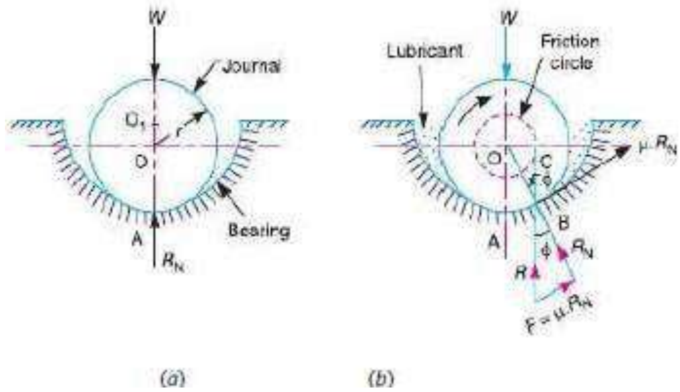
and the torque required to lower the load

$$T = P \cdot \frac{d}{2} = W \tan(\phi - \alpha)\frac{d}{2}$$

In the above expression, if $\phi < \alpha$, then torque required to lower the load will be ***negative.*** In other words, the load will start moving downward without the application of any torque. Such a condition is known as ***over hauldingof screws.*** If however, $\phi > \alpha$, the torque required to lower the load will ***positive***, indicating that an effort is applied to lower the load. Such a screw is known as ***self locking screw.*** In other words, a screw will be self locking if the friction angle is greater than helix angle or coefficient of friction is greater than tangent of helixangle
i.e. $\propto$ or $\tan \phi > \tan \alpha$.

Friction in Journal Bearing-Friction Circle

A journal bearing forms a turning pair as shown in Fig. 10.15 (*a*). The fixed outer element of a turning pair is called a ***bearing***and that portion of the inner element (*i.e.* shaft) which fits in the bearing is called a ***journal.*** The journal isslightlylessindiameterthanthebearing,inordertopermitthefreemovementofthejournalinabearing.



When the bearing is not lubricated (or the journal is stationary), then there is a line contact between the two elements as shown in Fig. 10.15 (*a*). The load W on the journal and normal reaction R_N (equal to W) of the bearing acts through the centre. The reaction R_N acts vertically upwards at point A. This point $A$is known as ***seat*** or ***point ofpressure.***

Now consider a shaft rotating inside a bearing in clockwise direction as shown in Fig. 10.15 (*b*). The lubricant between the journal and bearing forms a thin layer which gives rise to a greasy friction.Therefore, the reaction R does not act vertically upward, but acts at another point of pressure B. This is due to the fact thatwhen shaft rotates, a frictional force $F = \mu R_N$ acts at the circumference of the shaft which has a tendency to rotate the shaft in opposite direction of motion and this shifts the point A to point B.

In order that the rotation may be maintained, there must be a couple rotating the shaft

.Let ϕ = Angle between R (resultant of F and R_N) and R_N,

μ = Coefficient of friction between the journal and bearing,

T = Frictional torque in N-m, and

r = Radius of the shaft in metres.

For uniform motion, the resultant force acting on the shaft must be zero and the resultant turning moment on the shaft must be zero. In other words,

$$R = W \text{, and } T = W \times OC = W \times OB \sin\phi = W.r\sin\phi$$

Since ϕ is very small, therefore substituting $\sin\phi = \tan\phi$

$$T = W.r\tan\phi = \mu.W.r$$

If the shaft rotates with angular velocity ω rad/s, then power wasted in friction,

$$P = T.\omega = T \times 2\pi N/60 \text{ watts}$$

Flat Pivot Bearing

When a vertical shaft rotates in a flat pivot bearing (known as **foot step bearing**), as shown in Fig. 10.17, the sliding friction will be along the surface of contact between the shaft and the bearing.

Let W = Load transmitted over the bearing surface,

R = Radius of bearing surface,

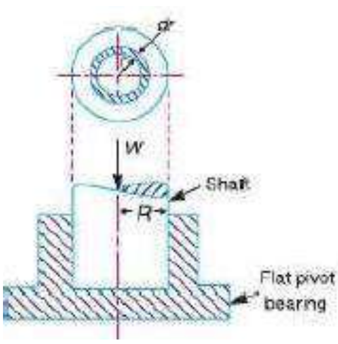
p = Intensity of pressure per unit area of bearing surface between rubbing surfaces, and

μ = Coefficient of friction.

We will consider the following two cases :

1. When there is a uniform pressure ; and

2. When there is a uniform wear.



1. *Considering unifrompressure*

When the pressure is uniformly distributed over the bearing area, then

$$p = \frac{W}{\pi R^2}$$

Consider a ring of radius r and thickness dr of the bearing area.

Area of bearing surface, $A = 2\pi r.dr$

Load transmitted to the ring,

$$\delta W = p \times A = p \times 2\pi r.dr$$

Frictional resistance to sliding on the ring acting tangentially at radius r,

$$F_r = \mu.\delta W = \mu p \times 2\pi r.dr = 2\pi\mu.p.r.dr$$

Frictional torque on the ring,

$$T_r = F_r \times r = 2\pi\mu p\, r.dr \times r = 2\pi\mu p\, r^2\, dr$$

Integrating this equation within the limits from 0 to R for the total frictional torque on the pivot bearing.

2. *Considering uniformwear*

We have already discussed that the rate of wear depends upon the intensity of pressure (p) and the velocity of rubbing surfaces (v). It is assumed that the rate of wear is proportional to the product of intensity of pressure and the velocity of rubbing surfaces (*i.e.* $p.v.$.). Since the velocity of rubbing surfaces increases with the distance (*i.e.* radius r) from the axis of the bearing, therefore for uniformwear

$$p.r = C \text{ (a constant)}$$

and the load transmitted to the ring,

$$\delta W = p \times 2\pi r.dr$$

∴ Total load transmitted to the bearing

$$W = \int_0^R 2\pi C.dr = 2\pi C[r]_0^R = 2\pi C.R \text{ or } C = \frac{W}{2\pi R}$$

We know that frictional torque acting on the ring,

$$T_r = 2\pi\mu p r^2 dr = 2\pi\mu \cdot \frac{C}{r} \cdot r^2 dr$$

$$= 2\pi\mu.C.r\, dr$$

Conical Pivot Bearing

The conical pivot bearing supporting a shaft carrying a load *W* is shown in Fig

P⁻$_0$= Intensity of pressure normal to thecone,

〈= Semi angle of the cone,

∝= Coefficient of friction between the shaft and the bearing, and

R= Radius of the shaft.

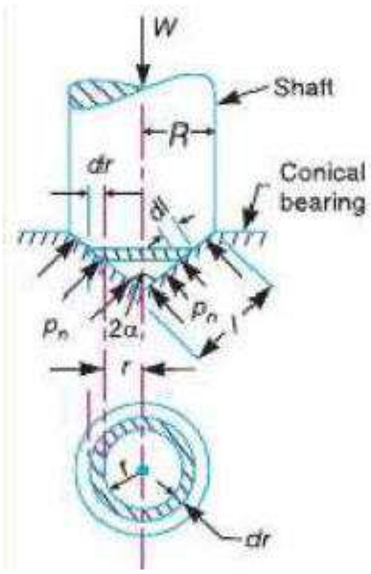
Consider a small ring of radius *r* and thickness *dr.* Let *dl* is the length of ring along the cone, such that

dl= *dr*cosec 〈

Area of the ring,

A = 2□*r*.*dl* = 2□*r*.*dr* cosec 〈

...(∵ *dl* = *dr*cosec 〈)



1. Considering uniformpressure

We know that normal load acting on the ring,

W $_n$= Normal pressure × Area

= p_n× 2□*r*.*dr* cosec 〈

and vertical load acting on the ring,

*™*W* = Vertical component of ™*W* $_n$= ™*W* $_n$.sin〈

=p_n×2□*r*.*dr*cosec〈.sin〈=p_n×2□*rdr* p_n=

W /□*R*$_2$

We know that frictional force on the ring acting tangentially at radius *r*,

$$F_r = \propto .™W_n = \propto .p_n.2□r.dr \text{ cosec } \langle = 2□\propto .\ p_n.\text{cosec }\langle .r.dr$$

and frictional torque acting on the ring,

$$T_r = F_r \cdot r = 2\ □\propto .\ p_n.\text{cosec }\langle .r.dr \cdot r = 2□\propto .\ p_n\text{cosec }\langle .r^2.dr$$

Integrating the expression within the limits from 0 to R for the total frictional torque on the conical pivot bearing

2. Considering uniformwear

In Fig., let $p_r$be the normal intensity of pressure at a distance r from the central axis. We know that, in case of uniform wear, the intensity of pressure varies inversely with thedistance

$$p_r.r = C \text{ (a constant) or } p_r = C/r$$

and the load transmitted to the ring,

$$™W = p_r \cdot 2□r.dr = \frac{C}{r} \cdot 2□r.dr = 2□C.dr$$

4 Total load transmitted to the bearing,

$$W = \int_0^R 2□C.dr = 2□C\,[r]_0^R = 2□C.R \text{ or } C = \frac{W}{2□R}$$

We know that frictional torque acting on the ring,

$$T_r = 2□\propto .\ p_r.\text{cosec }\langle .r^2.dr = 2□\propto \cdot \frac{C}{r} \cdot \text{cosec }\langle .r^2.dr$$

$$= 2□\propto .C.\text{cosec }\langle .r.dr$$

4 Total frictional torque acting on the bearing,

$$T = \int_0^R 2□\propto .C.\text{cosec }\langle .r.dr = 2□\propto .C.\text{cosec }\langle \left[\frac{r^2}{2}\right]_0^R$$

$$= 2□\propto .C.\text{cosec }\langle \cdot \frac{R^2}{2} = □\propto .C.\text{cosec }\langle .R^2$$

Substituting the value of C, we have

$$T = □\propto \cdot \frac{W}{2□R} \cdot \text{cosec }\langle R^2 = \frac{1}{2}\propto .W.R \text{ cosec }\langle = \frac{1}{2}\propto .W.l$$

1. Clutches

Friction Clutches

A friction clutch has its principal application in the transmission of power of shafts and machines which must be started and stopped frequently. Its application is also found in cases in which power is to be delivered to machines partially or fully loaded. The force of friction is used to start the driven shaft from rest and gradually brings it up to the proper speed without excessive slipping of the friction surfaces. In automobiles, friction clutch is used to connect the engine to the driven shaft. In operating such a clutch, care should be taken so that the friction surfaces engage easily and gradually brings the driven shaft up to proper speed. The proper alignment of the bearing must be maintained and it should be located as close to the clutch as possible. It maybe noted that

1. The contact surfaces should develop a frictional force that may pick up and hold the load with reasonably low pressure between the contactsurfaces.

2. The heat of friction should be rapidly dissipated and tendency to grab should be at aminimum.

3. The surfaces should be backed by a material stiff enough to ensure a reasonably uniform distribution of pressure.

The friction clutches of the following types are important from the subject point of view:

1. Disc or plate clutches (single disc or multiple discclutch),

2. Cone clutches,and

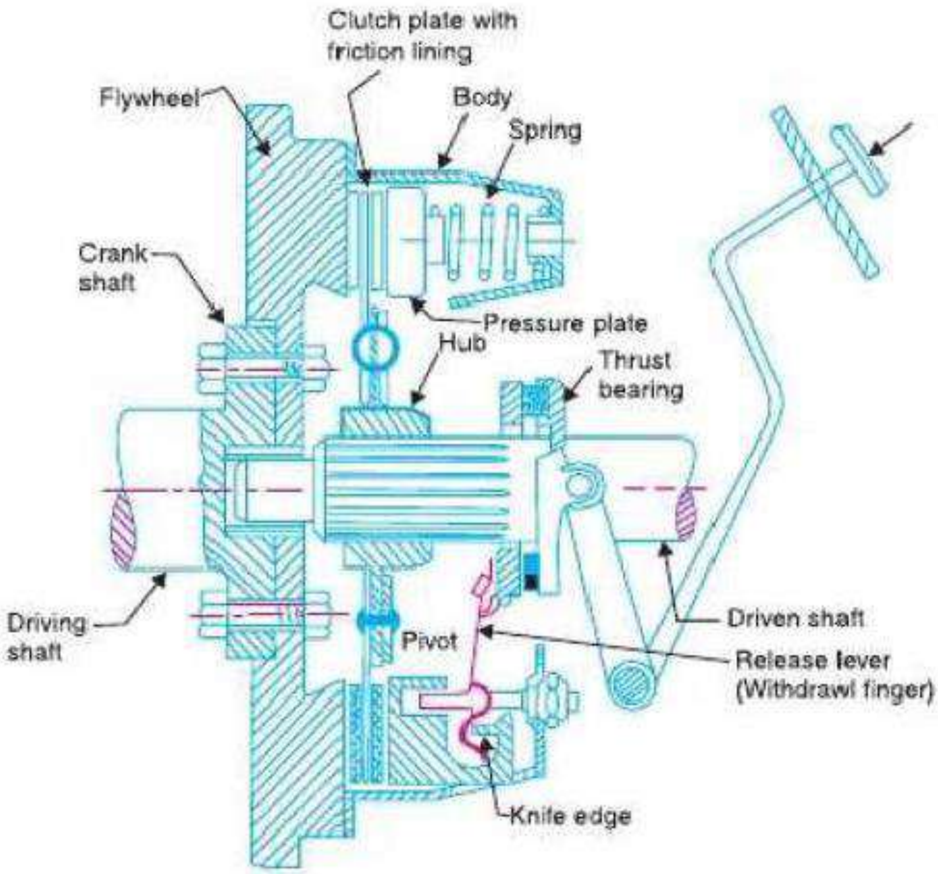
3. Centrifugalclutches.

We shall now discuss, these clutches, in detail, in the following pages. It may be noted that the disc and cone clutches are based on the same theory as the pivot and collarbearings.

Single Disc or Plate Clutch

A single disc or plate clutch, as shown in Fig. 10.21, consists of a clutch plate whose both sides are faced with a friction material (usually of Ferrodo). It is mounted on the hub which is free to move axially along the splines of the driven shaft. The pressure plate is mounted inside the clutch body which is bolted to the flywheel. Both the pressure plate and the flywheel rotate with the engine crankshaft or the driving shaft. The pressure plate pushes the clutch plate towards the flywheel by a set of strong springs which are arranged radially inside the body. The three levers (also known as release levers or fingers) are carried on pivots suspended from the case of the body. These are arranged in such a manner so that the pressure plate moves away from the flywheel by the inward movement of a thrustbearing.Thebearingismounteduponaforkedshaftandmovesforwardwhentheclutchpedalispressed.

When the clutch pedal is pressed down, its linkage forces the thrust release bearing to move in towards the flywheel and pressing the longer ends of the levers inward. The levers are forced to turn on their suspended pivot and the pressure plate moves away from the flywheel by the knife edges, thereby compressing the clutch springs. This action removes the pressure from the clutch plate and thus moves back from the flywheel and the driven shaft becomes stationary. On the other hand, when the foot is taken off from the clutch pedal, the thrust bearing moves back by the levers. This allows the springs to extend and thus the pressure plate pushes the clutch plate back towards the flywheel.



The axial pressure exerted by the spring provides a frictional force in the circumferential direction when the relative motion between the driving and driven members tends to take place. If the torque due to this frictional force exceeds the torque to be transmitted, then no slipping takes place and the power is transmitted from the driving shaft to the drivenshaft.

Now consider two friction surfaces, maintained in contact by an axial thrust W, as shown in Fig. (*a*).

T = Torque transmitted by the clutch

p = Intensity of axial pressure with which the contact surfaces are held together,

r_1 and r_2 = External and internal radii of friction faces, and

$\propto$ = Coefficient of friction.

Consider an elementary ring of radius r and thickness dr as shown in Fig. (*b*).

We know that area of contact surface or friction surface,

= 2 □$r.dr$

Normal or axial force on the ring,

™W = Pressure × Area = p × 2 □$r.dr$

and the frictional force on the ring acting tangentially at radius r,

$F_r = \propto.\text{™}W = \propto.p \times 2\,\square r.dr$

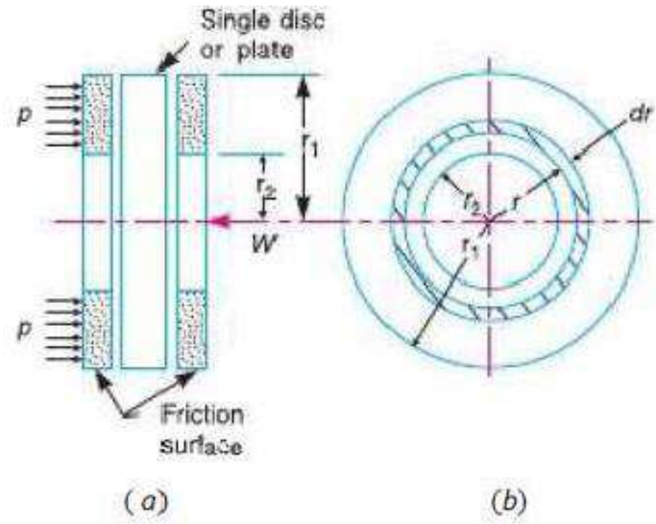
Frictional torque acting on the ring,

$T_r = F_r \times r = \propto.p \times 2\,\square r.dr \times r = 2\,\square \times \propto .p.r^2\, dr$

We shall now consider the following two cases :

1. When there is a uniform pressure,and

2. When there is auniform wear.



1. *Considering uniform pressure*

When the pressure is uniformly distributed over the entire area of the friction face, then the intensity ofpressure,

$$p = \frac{W}{\square[(r_1)^2 - (r_2)^2]}$$

We have discussed above that the frictional torque on the elementary ring of radius r and thickness $dr$is

Integrating this equation within the limits from r_2 to r_1 for the total frictional torque.

4 Total frictional torque acting on

$$T = \int_{r_1}^{r_2} 2\,\square \propto p.r^2.dr$$

Substituting the value of p from e

$$T = 2\,\square \propto \cdot \frac{}{\square[(r_1)}$$

$T_r = 2\,\square \propto.p.r^2\, dr$

$$= \frac{2}{3} \cdot \propto.W \left| \frac{(r}{\square (} \right.$$

R = Mean radi

$$= \frac{2}{3} \left| \frac{(r_1)^3 -}{\square (r_1)^2 -} \right.$$

2. *Considering uniformwear*

In Fig. 10.22, let p be the normal intensity of pressure at a distance r from the axis of the clutch. Since the intensity of pressure varies inversely with the distance,therefore

$$p.r = C \text{ (a constant) or } p = C/r \qquad \ldots(i)$$

and the normal force on the ring,

$$\delta W = p.2\pi r.dr = \frac{C}{r} \cdot 2\pi C.dr = 2\pi C.dr$$

∴ Total force acting on the friction surface,

$$W = \int_{r_2}^{r_1} 2\pi C\,dr = 2\pi C\,[r]_{r_2}^{r_1} = 2\pi C\,(r_1 - r_2)$$

or

$$C = \frac{W}{2\pi (r_1 - r_2)}$$

We know that the frictional torque acting on the ring,

$$T_r = 2\pi\mu.\,p\,r^2.dr = 2\pi\mu \cdot \frac{C}{r}\, r^2.dr = 2\pi\mu.C.r.dr \qquad \ldots(\because p = C/r)$$

∴ Total frictional torque on the friction surface,

$$T = \int_{r_2}^{r_1} 2\pi\mu.C.r.dr = 2\pi\mu.C\left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi\mu.C\left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi\mu.C\,[(r_1)^2 - (r_2)^2] = \pi\mu \cdot \frac{W}{2\pi(r_1 - r_2)}\,[(r_1)^2 - (r_2)^2]$$

$$= \frac{1}{2} \cdot \mu.W(r_1 + r_2) = \mu.W.R$$

where

$$R = \text{Mean radius of the friction surface} = \frac{r_1 + r_2}{2}$$

Multiple Disc Clutch

A multiple disc clutch, as shown in Fig. 10.23, may be used when a large torque is to be transmitted. The inside discs (usually of steel) are fastened to the driven shaft to permit axial motion (except for the last disc). The outside discs (usually of bronze) are held by bolts and are fastened to the housing which is keyed to the driving shaft. The multiple disc clutches are extensively used in motor cars, machine toolsetc.

Let

n_1 = Number of discs on the driving shaft, and

n_2 = Number of discs on the driven shaft.

4 Number of pairs of contact surfaces,

$$n = n_1 + n_2 - 1$$

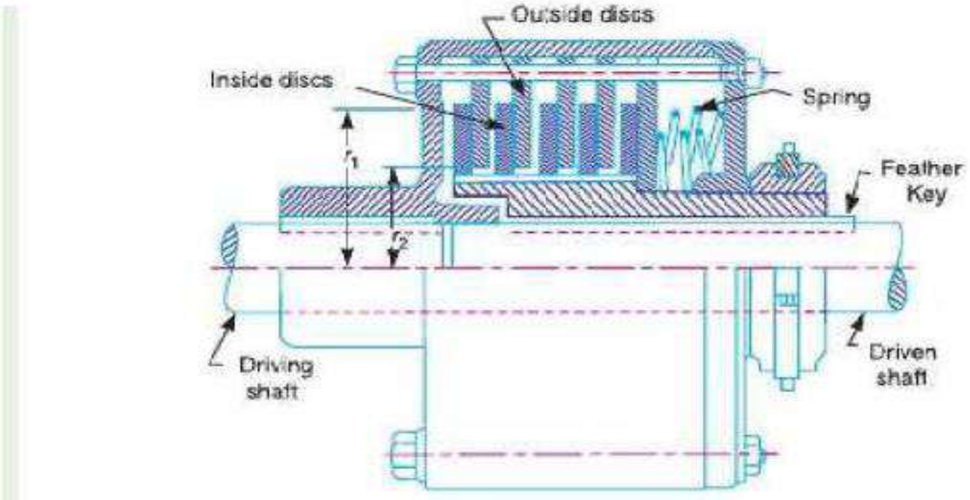
and total frictional torque acting on the friction surfaces or on the clutch,

$$T = n.\mu.W.R$$

where R = Mean radius of the friction surfaces

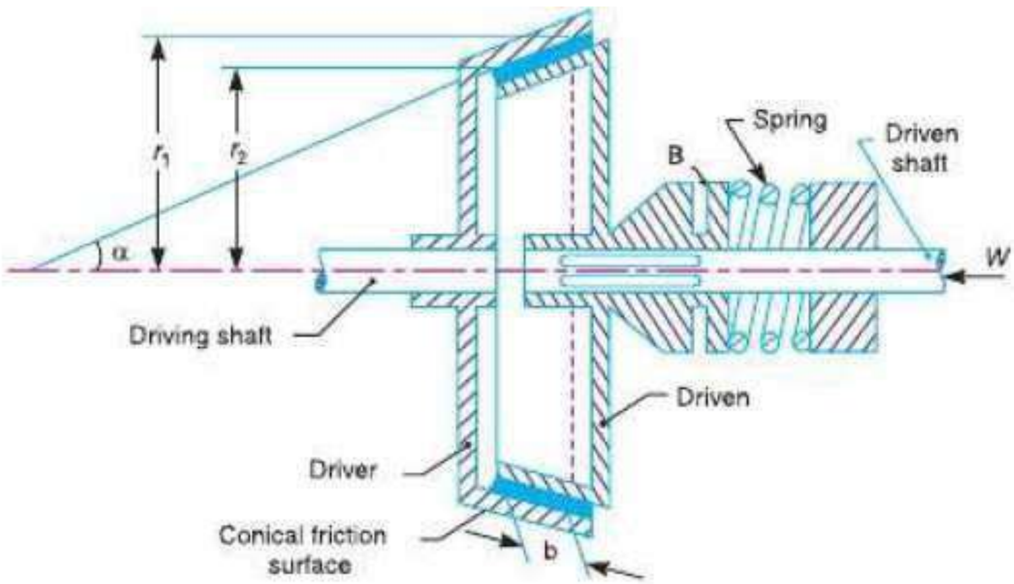
$$= \frac{2}{3}\left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right] \quad \text{...(For uniform pressure)}$$

$$= \frac{r_1 + r_2}{2} \quad \text{...(For uniform wear)}$$

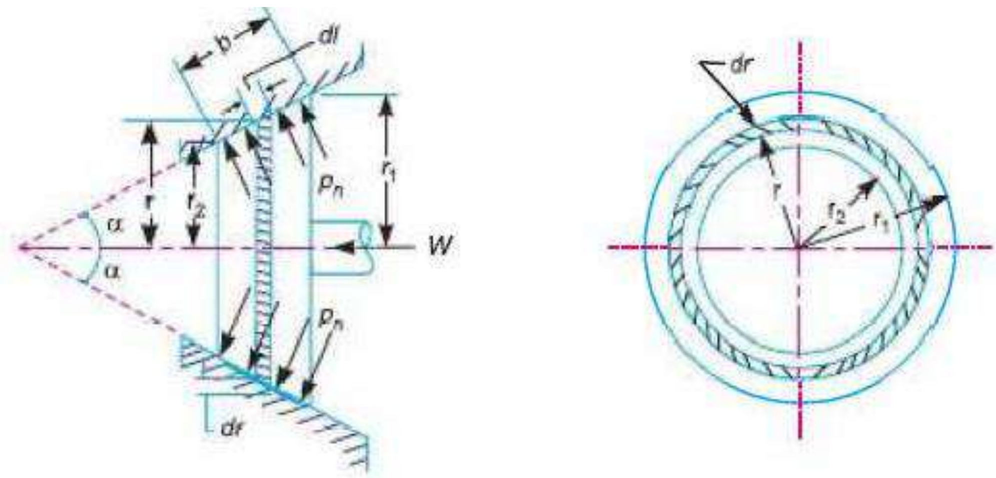


Cone Clutch

A cone clutch, as shown in Fig. 10.24, was extensively used in automobiles but now-a-days it has been replaced completely by the disc clutch



It consists of one pair of friction surface only. In a cone clutch, the driver is keyed to the driving shaft by a sunk key and has an inside conical surface or face which exactly fits into the outside conical surface of the driven. The driven member resting on the feather key in the driven shaft, may be shifted along the shaft by a forked lever provided at *B*, in order to engage the clutch by bringing the two conical surfaces in contact. Due to the frictional resistance set up at this contact surface, the torque is transmitted from one shaft to another. In some cases, a spring is placed around the driven shaft in contact with the hub of the driven. This spring holds the clutch faces in contact and maintains the pressure between them, and the forked lever is used only for disengagement of the clutch. The contact surfaces of the clutch may be metal to metal contact, but more often the driven member is lined with some material like wood, leather, cork or asbestos etc. The material of the clutch faces (*i.e.* contact surfaces) depends upon the allowable normal pressure and the coefficient of friction. Consider a pair of friction surface as shown in Fig. 10.25 (*a*). Since the area of contact of a pair of friction surface is a frustrum of a cone, therefore the torque transmitted by the cone clutch may be determined in the similar manner as discussed for conical pivot bearings inArt.



(*a*) (*b*)

p_n= Intensity of pressure with which the conical friction surfaces are held together (*i.e.* normal pressure between contactsurfaces),

$r_1$and r_2 = Outer and inner radius of friction surfaces respectively.

R = Mean radius of the friction surface

〈 = Semi angle of the cone (also called face angle of the cone) or the angle of the friction surface with the axis of the clutch,

∝ = Coefficient of friction between contact surfaces, and

b = Width of the contact surfaces (also known as face width or clutch face).

Consider a small ring of radius *r* and thickness *dr*, as shown in Fig. 10.25 (*b*). Let *dl* is length of ring of the friction surface, such that

$$dl = dr.\text{cosec}\,\alpha$$

Area of the ring,

$$A = 2\pi r.dl = 2\pi r.dr\ \text{cosec}\,\alpha$$

We shall consider the following two cases :

1. When there is a uniform pressure,and

2. When there is a uniformwear.

1. Considering uniformpressure

We know that normal load acting on the ring,

$$W_n = \text{Normal pressure} \times \text{Area of ring} = p_n \times 2\pi r.dr.\text{cosec}\,\alpha$$

and the axial load acting on the ring,

$$W = \text{Horizontal component of } \delta W_n (\textit{i.e.} \text{ in the direction of } W)$$

$$= \delta W_n \times \sin\alpha = p_n \times 2\pi r.dr.\ \text{cosec}\,\alpha \times \sin\alpha = 2\pi \times p_n.r.dr$$

Total axial load transmitted to the clutch or the axial spring force required,

$$W = \int_{r_2}^{r_1} 2\pi p_n r.dr = 2\pi p_n \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi p_n \left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi p_n [(r_1)^2 - (r_2)^2]$$

$$p_n = \frac{W}{\pi[(r_1)^2 - (r_2)^2]}$$

$$W = \int_{r_2}^{r_1} 2\pi p_n r.dr = 2\pi p_n \left[\frac{r^2}{2}\right]_{r_2}^{r_1} = 2\pi p_n \left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \pi p_n [(r_1)^2 - (r_2)^2]$$

$$p_n = \frac{W}{\pi[(r_1)^2 - (r_2)^2]}$$

We know that frictional force on the ring acting tangentially at radius r,

$$F_r = \mu.\delta W_n = \mu.p_n \times 2\pi r.dr.\text{cosec}\,\alpha$$

Frictional torque acting on the ring,

$$T_r = F_r \times r = \mu.p_n \times 2\pi r.dr.\ \text{cosec}\,\alpha.r = 2\pi\mu.p_n.\text{cosec}\,\alpha.r^2\,dr$$

Integrating this expression within the limits from r_2 to r_1 for the total frictional torque on the clutch.

Total frictional torque,

$$T = \int_{r_2}^{r_1} 2\pi\mu . p_n .\text{cosec}\,\alpha . r^2 .dr = 2\pi\mu\, p_n .\text{cosec}\,\alpha \left[\frac{r^3}{3}\right]_{r_2}^{r_1}$$

$$= 2\pi\mu\, p_n .\text{cosec}\,\alpha \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

Substituting the value of p_n from equation ***(i)***, we get

$$T = 2\pi\mu \cdot \frac{W}{\pi[(r_1)^2 - (r_2)^2]} \cdot \text{cosec}\,\alpha \left[\frac{(r_1)^3 - (r_2)^3}{3}\right]$$

$$= \frac{2}{3} \cdot \mu . W .\text{cosec}\,\alpha \left[\frac{(r_1)^3 - (r_2)^3}{(r_1)^2 - (r_2)^2}\right]$$

2. Considering uniformwear

In Fig. 10.25, let p_r be the normal intensity of pressure at a distance r from the axis of the clutch. We know that, in case of uniform wear, the intensity of pressure varies inversely with the distance.

$\therefore \quad p_r . r = C$ (a constant) or $\quad p_r = C / r$

We know that the normal load acting on the ring,

$$\delta W_n = \text{Normal pressure} \times \text{Area of ring} = p_r \times 2\pi r.dr\,\text{cosec}\,\alpha$$

and the axial load acting on the ring ,

$$\delta W = \delta W_n \times \sin\alpha = p_r . 2\pi r.dr.\text{cosec}\,\alpha . \sin\alpha = p_r \times 2\pi r.dr$$

$$= \frac{C}{r} \cdot 2\pi r.dr = 2\pi C.dr \qquad ...(\because p_r = C / r)$$

$\therefore$ Total axial load transmitted to the clutch,

$$W = \int_{r_2}^{r_1} 2\pi C\, dr = 2\pi C [r]_{r_2}^{r_1} = 2\pi C (r_1 - r_2)$$

or $$C = \frac{W}{2\pi(r_1 - r_2)} \qquad ...\textit{(iii)}$$

We know that frictional force acting on the ring,

$$F_r = \mu . \delta W_n = \mu . p_r \times 2\pi r \times dr\,\text{cosec}\,\alpha$$

and frictional torque acting on the ring,

$$T_r = F_r \times r = \mu\, p_r \times 2\pi r.dr.\text{cosec}\,\alpha \times r$$

$$= \mu \frac{C}{r} \cdot 2\pi r^2 .dr.\text{cosec}\,\alpha = 2\pi\mu . C\,\text{cosec}\,\alpha \cdot r\, dr$$

4 Total frictional torque acting on the clutch,

$$T = \int_{r_2}^{r_1} 2\pi\mu . C.\text{cosec}\,\alpha . r\,dr = 2\pi\mu . C.\text{cosec}\,\alpha \left[\frac{r^2}{2}\right]_{r_2}^{r_1}$$

$$= 2\pi\mu . C.\text{cosec}\,\alpha \left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

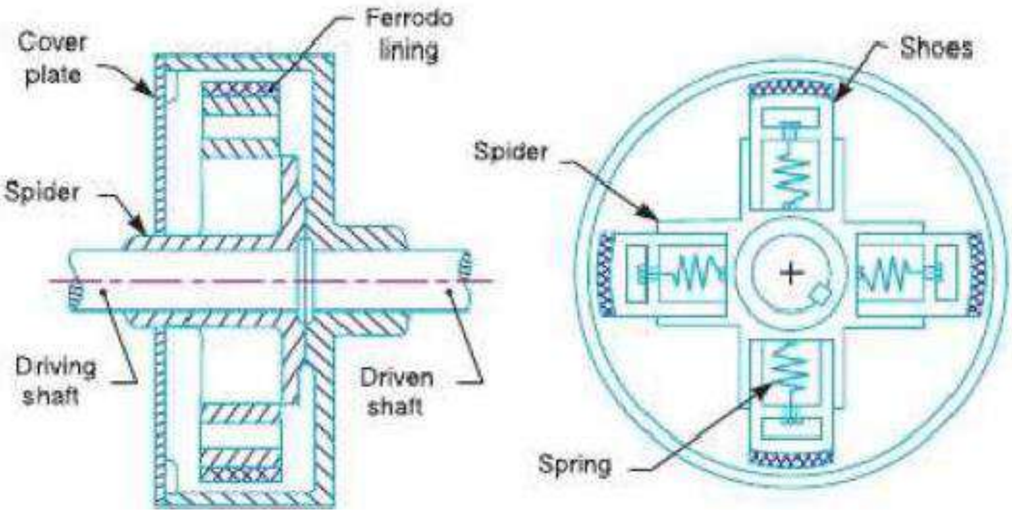
Substituting the value of C from equation (*i*), we have

$$T = 2\pi\mu \cdot \frac{W}{2\pi(r_1 - r_2)} \cdot \text{cosec}\,\alpha \left[\frac{(r_1)^2 - (r_2)^2}{2}\right]$$

$$= \mu . W\,\text{cosec}\,\alpha \left(\frac{r_1 + r_2}{2}\right) = \mu . W . R\,\text{cosec}\,\alpha$$

Centrifugal Clutch

The centrifugal clutches are usually incorporated into the motor pulleys. It consists of a number of shoes on the inside of a rim of the pulley, as shown in Fig. 10.28. The outer surface of the shoes are covered with a friction material. These shoes, which can move radially in guides, areheld



against the boss (or spider) on the driving shaft by means of springs. The springs exert a radially inward force which is assumed constant. The mass of the shoe, when revolving, causes it to exert a radially outward force (*i.e.* centrifugal force). The magnitude of this centrifugal force depends upon the speed at which the shoe is revolving. A little consideration will show that when the centrifugal force is less than the spring force, the shoe remains in the same position as when the driving shaft was stationary, but when the centrifugal force is equal to the spring force, the shoe is just floating. When the centrifugal force exceeds the spring force, the shoe moves outward and comes into contact with the driven member and presses against it. The force with which the shoe presses against the driven memberisthedifferenceofthecentrifugalforceandthespringforce.Theincreaseofspeedcausestheshoetopress harder

and enables more torque to be transmitted.

In order to determine the mass and size of the shoes, the following procedure is adopted :

1. Mass of theshoes

Consider one shoe of a centrifugal clutch as shown in Fig

Let m = Mass of each shoe,

n = Number of shoes,

r = Distance of centre of gravity of the shoe from the centre of the spider,

R = Inside radius of the pulley rim,

N = Running speed of the pulley in r.p.m.,

ω = Angular running speed of the pulley in rad/s = 2□N/60 rad/s,

ω_1 = Angular speed at which the engagement begins to take place, and

$\propto$ = Coefficient of friction between the shoe and rim.

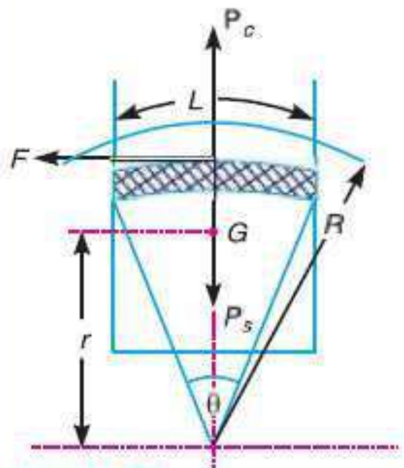


Fig. 10.29. Forces on a shoe of centrifugal clutch.

We know that the centrifugal force acting on each shoe at the running speed,

$$^*P_c = m.\omega^2.r$$

We know that the centrifugal force acting on each shoe at the running speed,

$$^*P_c = m.\omega^2.r$$

and the inward force on each shoe exerted by the spring at the speed at which engagement begins to take place,

$$P_s = m\,(\omega_1)^2\,r$$

The net outward radial force (*i.e.* centrifugal force) with which the shoe presses against the rim at the running speed

$$= P_c - P_s$$

and the frictional force acting tangentially on each shoe,

$$F = \propto (P_c - P_s)$$

Frictional torque acting on each shoe,

$$= F \times R = \propto (P_c - P_s)\,R$$

and total frictional torque transmitted,

$$T = \propto (P_c - P_s)\, R \times n = n.F.R$$

From this expression, the mass of the shoes (m) may be evaluated.

2. Size of theshoes

l = Contact length of the shoes,

b = Width of the shoes,

R = Contact radius of the shoes. It is same as the inside radius of the rim of the pulley.

θ = Angle subtended by the shoes at the centre of the spider in radians.

p = Intensity of pressure exerted on the shoe. In order to ensure reasonable life, the intensity of pressure may be taken as 0.1N/mm2

Area of contact of theshoe,

$$A = l.b$$

and the force with which the shoe presses against the rim

$$= A \times p = l.b.p$$

Since the force with which the shoe presses against the rim at the running speed is ($P_c - P_s$), therefore

$$l.b.p = P_c - P_s$$

From this expression, the width of shoe (b) may be obtained.

Brakes and Dynamometers

Introduction

A ***brake*** is a device by means of which artificial frictional resistance is applied to a moving machine member, in order to retard or stop the motion of a machine. In the process of performing this function, the brake absorbs either kinetic energy of the moving member or potential energy given up by objects being lowered by hoists, elevators etc. The energy absorbed by brakes is dissipated in the form of heat. This heat is dissipated in the surrounding air (or water which is circulated through the passages in the brake drum) so that excessive heating of the brake lining does not take place. The capacity of a brake depends upon the following factors:

1. The unit pressure between the brakingsurfaces,
2. The coefficient of friction between the brakingsurfaces,
3. The peripheral velocity of the brakedrum,
4. The projected area of the friction surfaces,and
5. The ability of the brake to dissipate heat equivalent to the energy beingabsorbed.

The major functional difference between a clutch and a brake is that a clutch is used to keep the driving and driven member moving together, whereas brakes are used to stop a moving member or to control its speed.

Materials for Brake Lining

The material used for the brake lining should have the following characteristics

1. It should have high coefficient of friction with minimum fading. In other words, the coeffi- cientof friction should remain constant with change intemperature.
2. It should have low wearrate.
3. It should have high heatresistance.
4. It should have high heat dissipation capacity.
5. It should have adequate mechanicalstrength.
6. It should not be affected by moisture and oil.

The materials commonly used for facing or lining of brakes and their properties are shown in the following table.

Types of Brakes

The brakes, according to the means used for transforming the energy by the braking elements, are classified as :

1. Hydraulic brakes *e.g.* pumps or hydrodynamic brake and fluidagitator,
2. Electric brakes *e.g.* generators and eddy current brakes,and
3. Mechanicalbrakes.

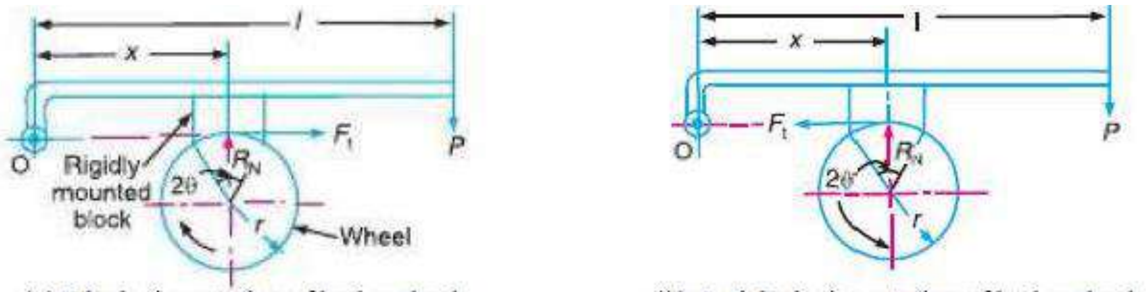
The hydraulic and electric brakes cannot bring the member to rest and are mostly used where large amounts of energy are to be transformed while the brake is retarding the load such as in laboratory dynamometers, high way trucks and electric locomotives. These brakes are also used for retarding or controlling the speed of a vehicle for down-hill travel. The mechanical brakes, according to the direction of acting force, may be divided into the following two groups:

(a) ***Radial brakes.*** In these brakes, the force acting on the brake drum is in radial direction. The radial brakes may be sub-divided into ***external brakes*** and ***internal brakes.*** According to the shape of the friction elements, these brakes may be ***block*** or ***shoe brakes*** and ***bandbrakes.***

(b) ***Axial brakes.*** In these brakes, the force acting on the brake drum is in axial direction. The axial brakes may be disc brakes and cone brakes. The analysis of these brakes is similar to clutches. Since we are concernedwithonly mechanicalbrakes,therefore,thesearediscussed,indetail,inthefollowingpages.

Single Block or Shoe Brake

A single block or shoe brake is shown in Fig. 19.1. It consists of a block or shoe which is pressed against the rim of a revolving brake wheel drum. The block is made of a softer material than the rim of the wheel. This type of a brake is commonly used on railway trains and tram cars. The friction between the block and the wheel causes a tangential braking force to act on the wheel, which retard the rotation of the wheel. The block is pressed against the wheel by a force applied to one end of a lever to which the block is rigidly fixed as shown in Fig. 19.1. The other end of the lever is pivoted on a fixed fulcrum*O*.



(*a*) Clockwise rotation of brake wheel (*b*) Anticlockwise rotation of brake wheel

If the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. In such cases, tangential braking force on thewheel,

$$F_t = \mu . R_N$$

and the braking torque, $T_B = F_t.r = \mu . R_N.r$

Let us now consider the following three cases :

Case 1. When the line of action of tangential braking force (F_t) passes through the fulcrum *O* of the lever, and the brake wheel rotates clockwise as shown in Fig. (*a*), then for equilibrium, taking moments about the fulcrum *O*, we have

$$R_N \cdot x = P \cdot l \text{ or } R_N = \frac{P \cdot l}{x}$$

Braking torque,

$$T_B = \propto.R_N.r = \propto \cdot \frac{P.l}{x} \cdot r = \frac{\propto.P.l.r}{x}$$

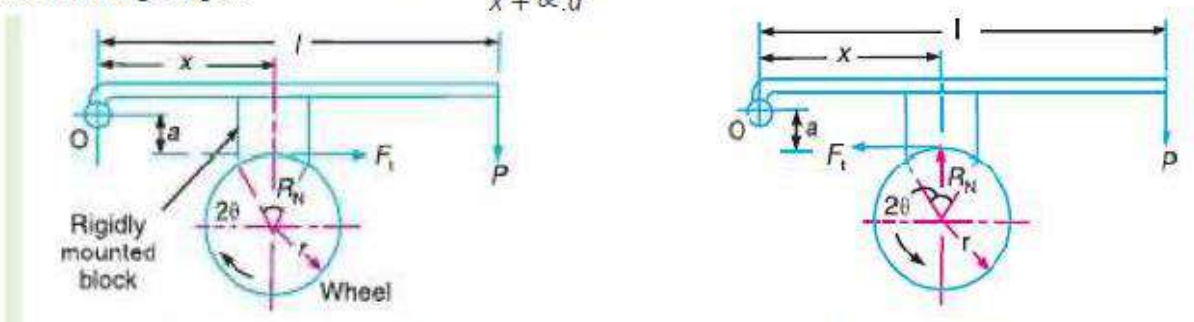
It may be noted that when the brake wheel rotates anticlockwise as shown in Fig. (*b*), then the braking torque is same,*i.e*

$$T_B = \propto.R_N.r = \frac{\propto.P.l.r}{x}$$

Case 2. When the line of action of the tangential braking force (*Ft*) passes through a distance '*a*' below the fulcrum *O*, and the brake wheel rotates clockwise as shown in Fig. (*a*), then for equilibrium, taking moments about the fulcrum*O*,

$$R_N \times x + F_t \times a = P.l \quad \text{or } R_N \times x + \mu R_N \times a = P.l \quad \text{or} \quad R_N = \frac{P.l}{x + \propto.a}$$

and braking torque, $T_B = \propto R_N \ r = \dfrac{\propto.\ p.l.r}{x + \propto.a}$



(*a*) Clockwise rotation of brake wheel. (*b*) Anticlockwise rotation of brake wheel.

When the brake wheel rotates anticlockwise, as shown in Fig. 19.2 (*b*), then for equilibrium.

$$R_N.x = P.l + F_t.a = P.l + \mu.R_N.a$$

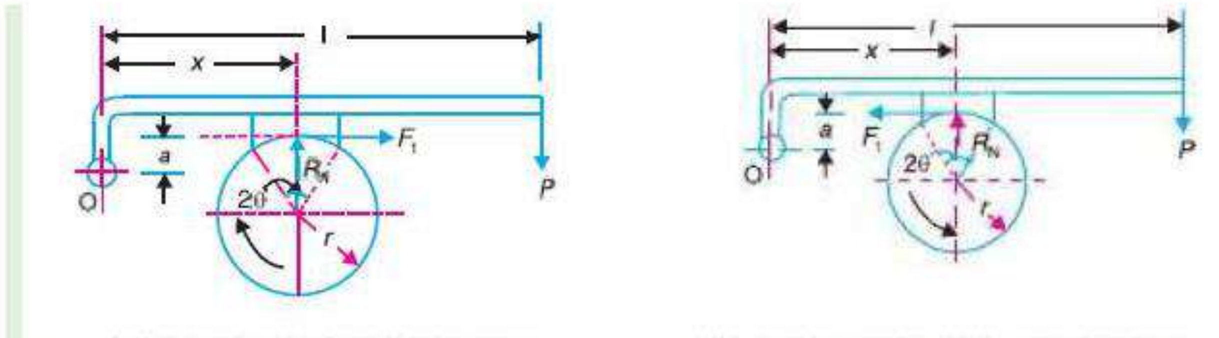
or $\quad R_N(x - \mu.a) = P.l \quad$ or $\quad R_N = \dfrac{P.l}{x - \propto.a}$

and braking torque, $T_B = \propto.R_N.r = \dfrac{\propto.P.l.r}{x - \propto.a}$

Case 3. When the line of action of the tangential braking force (*Ft*) passes through a distance '*a*' above the fulcrum *O*, and the brake wheel rotates clockwise as shown in Fig. 19.3 (*a*), then for equilibrium, taking moments about the fulcrum *O*, wehave

$$R_N.x = P.l + F_t \,.\, a = P.l + \mu.R_N.a$$

or $\quad R_N\,(x - \mu.a) = P.l \quad$ or $\quad R_N = \dfrac{P.l}{x - \propto.a}$



(*a*) Clockwise rotation of brake wheel. (*b*) Anticlockwise rotation of brake wheel.

and braking torque,

$$T_B = \mu.R_N.r = \frac{\propto.P.l.r}{x - \propto.a}$$

When the brake wheel rotates anticlockwise as shown in Fig. 19.3 (*b*), then for equilibrium, taking moments about the fulcrum *O*, wehave

$$R_N \times x + F_t \times a = P.l \quad \text{or} \quad R_N \times x + \mu.R_N \times a = P.l \text{ or } R_N = \frac{P.l}{x + \propto.a}$$

and braking torque, $\quad T_B = \mu.R_N.r = \dfrac{\propto.P.l.r}{x + \propto.a}$

Pivoted Block or Shoe Brake

We have discussed in the previous article that when the angle of contact is less than 60°, then it may be assumed that the normal pressure between the block and the wheel is uniform. But when the angle of contact is greater than 60°, then the unit pressure normal to the surface of contact is less at the ends than at the centre. In such cases, the block or shoe is pivoted to the lever, as shown in Fig. 19.4, instead of being rigidly attached to the lever. This gives uniform wear of the brake lining in the direction of the applied force. The braking torque for a pivoted block or shoe brake (*i.e.* when $2\theta > 60°$)is

given by

$$T_B = F_t \cdot r = \propto 2.R_N.r$$

where $\quad \propto 2$ = Equivalent coefficient of friction = $\dfrac{4\propto \sin\theta}{2\theta + \sin 2\theta}$, and

μ = Actual coefficient of friction.

These brakes have more life and may provide a higher braking torque.

Simple Band Brake

A band brake consists of a flexible band of leather, one or more ropes,or a steel lined with friction material, which embraces a part of the circumference of the drum. A band brake, as shown in Fig., is called a ***simple band brake*** in which one end of the band is attached to a fixed pin or fulcrum of the lever while the other end is attached to the lever at a distance b from the fulcrum. When a force P is applied to the lever at C, the lever turns about the fulcrum pin O and tightens the band on the drum and hence the brakes are applied. The friction between the band andthedrumprovidesthebrakingforce.Theforce$P$ontheleverat$C$maybedeterminedasdiscussedbelow:

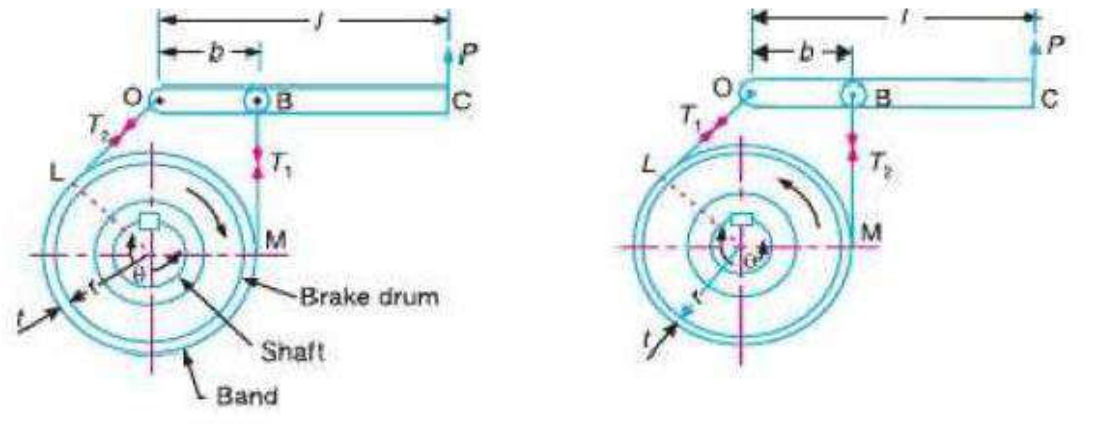
θ = Angle of lap (or embrace) of the band on the drum,

μ = Coefficient of friction between the band and the drum,

r = Radius of the drum,

t = Thickness of the band, and

r_e= Effective radius of the drum



(*a*) Clockwise rotation of drum. (*b*) Anticlockwise rotation of drum.

We know that limiting ratio of the tensions is given by the relation,

$$\frac{T_1}{T_2} = e^{\mu\theta} \quad \text{or} \quad 2.3\log\left(\frac{T_1}{T_2}\right) = \mu.\theta$$

and braking force on the drum = $T_1 - T_2$

Braking torque on the drum,

$T_B = (T_1 - T_2)\, r$ . . . (Neglecting thickness ofband)

$= (T_1 - T_2) r_e$ . . . (Considering thickness ofband)

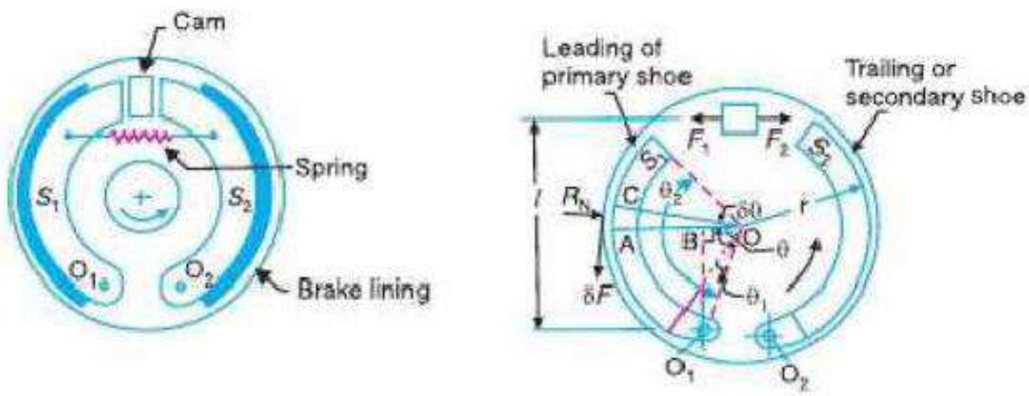
Now considering the equilibrium of the lever OBC. It may be noted that when the drum rotates in the clockwise direction, as shown in Fig.(a), the end of the band attached to the fulcrum O will be slack with tension T_2 and end of the band attached to B will be tight with tension T_1. On the other hand, when the drumrotates in the anticlockwise direction, as shown in Fig.(*b*), the tensions in the band will reverse, *i.e.* the end of the band attached to the fulcrum *O* will be tight with tension T_1 and the end of the band attached to *B* will be slack with tension T_2. Now taking moments about the fulcrum *O*, wehave

$P.l = T_1.b$ . . . (For clockwise rotation of thedrum)

$P.l = T_2.b$ . . . (For anticlockwise rotation of thedrum)

Internal Expanding Brake

An internal expanding brake consists of two shoes S_1 and S_2 as shown in Fig.. The outer surface of the shoes are lined with some friction material (usually with Ferodo) to increase the coefficient of friction and to prevent wearing away of the metal. Each shoe is pivoted at one end about a fixed fulcrum O_1 and O_2 and made to contact a cam at the other end. When the cam rotates, the shoes are pushed outwards against the rim of the drum. The friction betweentheshoesandthedrumproducesthebrakingtorqueandhencereducesthespeedofthedrum.Theshoesare



normally held in off position by a spring as shown in Fig. 19.24. The drum encloses the entire mechanism to keep out dust and moisture. This type of brake is commonly used in motor cars and lighttrucks.

Weshallnowconsidertheforcesactingonsuchabrake,whenthedrumrotatesintheanticlockwisedirection asshowninFig.19.25.Itmaybenotedthatfortheanticlockwisedirection,thelefthandshoeisknownas ***leading*** or ***primaryshoe*** whiletherighthandshoeisknown as ***trailing*** or ***secondaryshoe.***

Let r = Internal radius of the wheel rim,

b = Width of the brake lining,

p_1 = Maximum intensity of normal pressure,

p_N = Normal pressure,

F_1 = Force exerted by the cam on the leading shoe, and

F_2 = Force exerted by the cam on the trailing shoe.

Consider a small element of the brake lining *AC* subtending an angle $\delta\theta$ at the centre. Let *OA* makes an angle θ with OO_1 as shown in Fig. 19.25. It is assumed that the pressure distribution on the shoe is nearly uniform, however the friction lining wears out more at the free end. Since the shoe turns about O_1, therefore the rate of wear of the shoe lining at *A* will be proportional to the radial displacement of that point. The rate of wear of the shoe lining varies directly as the perpendicular distance from O_1 to *OA*, *i.e.* O_1B. From the geometry of the figure,

$$O_1B = OO_1 \sin\theta$$

and normal pressure at *A*,

$$p_N \quad \sin\theta \text{ or } p_N = p_1 \sin\theta$$

$$\delta R_N = \text{Normal pressure} \times \text{Area of the element}$$

$$= p_N\,(b.r.\delta\theta) = p_1 \sin\theta\,(b.r.\delta\theta)$$

and braking or friction force on the element,

$$\delta F = \mu \cdot \delta R_N = \mu .\ p_1 \sin\theta\,(b.r.\delta\theta)$$

∴ Braking torque due to the element about O,

$$\delta T_B = \delta F \cdot r = \mu .\ p_1 \sin\theta\,(b.r.\delta\theta)\,r = \mu .\ p_1 b\, r^2 (\sin\theta .\delta\theta)$$

and total braking torque about O for whole of one shoe,

$$T_B = \mu\, p_1 b\, r^2 \int_{\theta_1}^{\theta_2} \sin\theta\, d\theta = \mu\, p_1 b\, r^2 \left[-\cos\theta\right]_{\theta_1}^{\theta_2}$$

$$= \mu\, p_1 b r^2 (\cos\theta_1 - \cos\theta_2)$$

Moment of normal force δR_N of the element about the fulcrum O_1,

$$\delta M_N = \delta R_N \cdot O_1B = \delta R_N\,(OO_1 \sin\theta)$$

$$= p_1 \sin\theta\,(b.r.\delta\theta)\,(OO_1 \sin\theta) = p_1 \sin^2\theta\,(b.r.\delta\theta)\,OO_1$$

∴ Total moment of normal forces about the fulcrum O_1,

$$= p_1.b.r.OO_1 \int_{\theta_1}^{\theta_2} \frac{1}{2}(1 - \cos 2\theta)\,d\theta \qquad \ldots \left[\because \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)\right]$$

$$= \frac{1}{2}\, p_1.b.r.OO_1 \left[\theta - \frac{\sin 2\theta}{2}\right]_{\theta_1}^{\theta_2}$$

$$= \frac{1}{2}\, p_1.b.r.OO_1 \left[\theta_2 - \frac{\sin 2\theta_2}{2} - \theta_1 + \frac{\sin 2\theta_1}{2}\right]$$

$$= \frac{1}{2}\, p_1.b.r.OO_1 \left[(\theta_2 - \theta_1) + \frac{1}{2}(\sin 2\theta_1 - \sin 2\theta_2)\right]$$

Normal force acting on the element,

Moment of frictional force $\mu \delta F$ about the fulcrum O_1,

$$\mu \delta M_F = \mu \delta F \cdot AB = \mu \delta F (r - OO_1 \cos\theta) \quad \ldots (\because AB = r - OO_1 \cos\theta)$$

$$= \mu\, p_1 \sin\theta\, (b.r.\delta\theta)\,(r - OO_1 \cos\theta)$$

$$= \mu.\, p_1.b.r\, (r \sin\theta - OO_1 \sin\theta \cos\theta)\,\delta\theta$$

$$= \mu.\, p_1.b.r \left(r \sin\theta - \frac{OO_1}{2} \sin 2\theta \right) \delta\theta \quad \ldots (\because 2\sin\theta\cos\theta = \sin 2\theta)$$

$\therefore$ Total moment of frictional force about the fulcrum O_1,

$$M_F = \mu\, p_1 b\, r \int_{\theta_1}^{\theta_2} \left(r \sin\theta - \frac{OO_1}{2} \sin 2\theta \right) d\theta$$

$$= \mu\, p_1 b\, r \left[-r \cos\theta + \frac{OO_1}{4} \cos 2\theta \right]_{\theta_1}^{\theta_2}$$

$$= \mu\, p_1 b\, r \left[-r \cos\theta_2 + \frac{OO_1}{4} \cos 2\theta_2 + r \cos\theta_1 - \frac{OO_1}{4} \cos 2\theta_1 \right]$$

$$= \mu\, p_1 b\, r \left[r(\cos\theta_1 - \cos\theta_2) + \frac{OO_1}{4} (\cos 2\theta_2 - \cos 2\theta_1) \right]$$

Now for leading shoe, taking moments about the fulcrum O_1,

$$F_1 \times l = M_N - M_F$$

and for trailing shoe, taking moments about the fulcrum O_2,

$$F_2 \times l = M_N + M_F$$

Types of Dynamometers

Following are the two types of dynamometers, used for measuring the brake power of an engine.

1. Absorption dynamometers,and
2. Transmissiondynamometers.

In the *absorption* ***dynamometers,*** the entire energy or power produced by the engine is absorbed by the friction resistances of the brake and is transformed into heat, during the process of measurement. But in the ***transmission dynamometers,*** the energy is not wasted in friction but is used for doing work. The energy or power produced by the engine is transmitted through the dynamometer to some other machines where the power developed is suitablymeasured.

Classification of Absorption Dynamometers

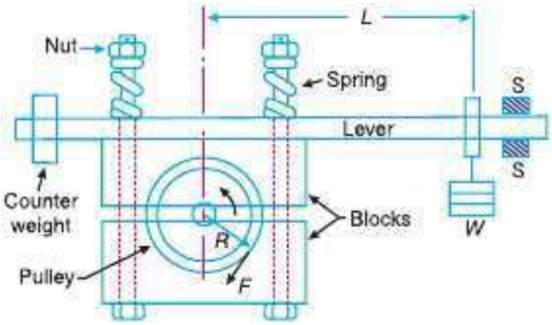
The following two types of absorption dynamometers are important from the subject point of view :

1. Prony brake dynamometer,and
2. Rope brakedynamometer.

These dynamometers are discussed, in detail, in the following pages.

Prony Brake Dynamometer

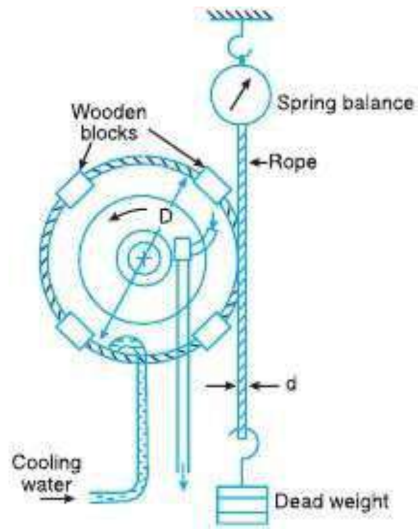
A simplest form of an absorption type dynamometer is a prony brake dynamometer, as shown in Fig. 19.31. It consists of two wooden blocks placed around a pulley fixed to the shaft of an engine whose power is required to be measured. The blocks are clamped by means of two bolts and nuts, as shown in Fig.. A helical spring is provided between the nut and the upper block to adjust the pressure on the pulley to control its speed. The upper block has a long lever attached to it and carries a weight *W* at its outer end. A counter weight is placed at the other end of the leverwhichbalancesthebrakewhenunloaded.TwostopsS,Sareprovidedtolimitthemotionofthelever



When the brake is to be put in operation, the long end of the lever is loaded with suitable weights *W* and the nuts are tightened until the engine shaft runs at a constant speed and the lever is in horizontal position. Under these conditions , the moment due to the weight *W* must balance the mo- ment of the frictional resistance between the blocks and the pulley.

Rope Brake Dynamometer

It is another form of absorption type dynamometer which is most commonly used for measur- ing the brake power of the engine. It consists of one, two or more ropes wound around the flywheel or rim of a pulley fixed rigidly to the shaft of an engine. The upper end of the ropes is attached to a spring balance while the lower end of the ropes is kept in position by applying a dead weight as shown in Fig.. In order to prevent the slipping of the rope over the flywheel, wooden blocks are placed at intervals around the circumference of the flywheel. In the operation of the brake, the engine is made to run at a constant speed. The frictional torque, due to the rope, must be equal to the torque being transmitted by the engine.



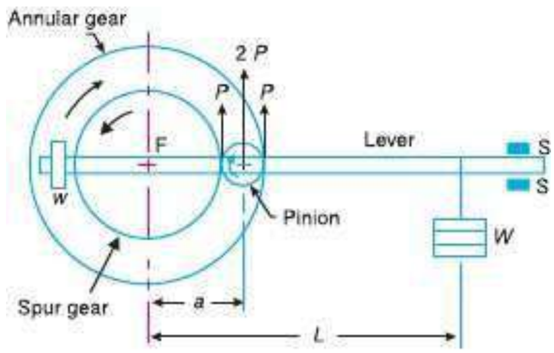
Classification of Transmission Dynamometers

The following types of transmission dynamometers are important from the subject point of view :

1. Epicyclic-traindynamometer,
2. Belt transmission dynamometer,and
3. Torsiondynamometer.

We shall now discuss these dynamometers, in detail, in the following pages.

Epicyclic-train Dynamometer



An epicyclic-train dynamometer, as shown in Fig. 19.33, consists of a simple epicyclic train of gears, *i.e.* a spur gear, an annular gear (a gear having internal teeth) and a pinion. The spur gear is keyed to the engine shaft (*i.e.* driving shaft) and rotates in anticlockwise direction. The annular gear is also keyed to the driving shaft and rotates in clockwise direction. The pinion or the intermediate gear meshes with both the spur and annular gears. The pinion revolves freely on a lever which is pivoted to the common axis of the driving and driven shafts. A weight $w$is placed at the smaller end of the lever in order to keep it in position. A little consideration will show that if the friction of the pin on which the pinion rotates is neglected, then the tangential effort P exerted by the spur gear on the pinion and the tangential reaction of the annular gear on the pinion areequal.

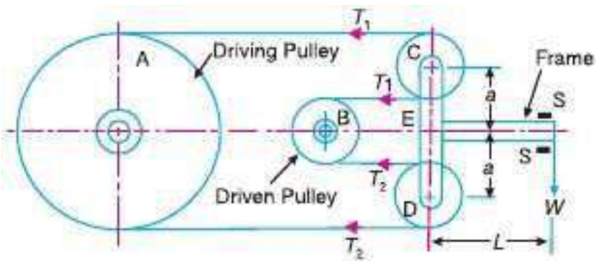
Since these efforts act in the upward direction as shown, therefore total upward force on the lever acting through the axis of the pinion is $2P$. This force tends to rotate the lever about its fulcrum and it is balanced by a dead weight W at the end of the lever. The stops S, S are provided to control the movement of the lever.

For equilibrium of the lever, taking moments about the fulcrum F,

$2P \times a = W.L$ or $P = W.L/2a$

Belt Transmission Dynamometer-Froude or Throneycroft Transmission Dynamometer

When the belt is transmitting power from one pulley to another, the tangential effort on the driven pulley is equal to the difference between the tensions in the tight and slack sides of the belt. A belt dynamometer is introduced to measure directly the difference between the tensions of the belt, while it isrunning.



A belt transmission dynamometer, as shown in Fig. 19.34, is called a Froude or Throneycroft transmission dynamometer. It consists of a pulley *A*(called driving pulley) which is rigidly fixed to the shaft of an engine whose power is required to be measured. There is another pulley *B* (called driven pulley) mounted on another shaft to which the power from pulley *A* is transmitted. The pulleys *A* and *B* are connected by means of a continuous belt passing round the two loose pulleys *C* and *D* which are mounted on a *T*-shaped frame. The frame is pivoted at *E* and its movement is controlled by two stops *S*,*S*. Since the tension in the tight side of the belt (T_1) is greater than the tension in the slack side of the belt (T_2), therefore the total force acting on the pulley *C* (*i.e.* $2T_1$) is greater than the total force acting on the pulley *D* (*i.e.* $2T_2$). It is thus obvious that the frame causes movement about *E* in the anticlockwise direction. In order to balance it, a weight *W* is applied at a distance *L* from *E* on the frame as shown inFig.

Now taking moments about the pivot *E*, neglecting friction,

$$2T_1 \cdot a = 2T_2 \cdot a + W .$$

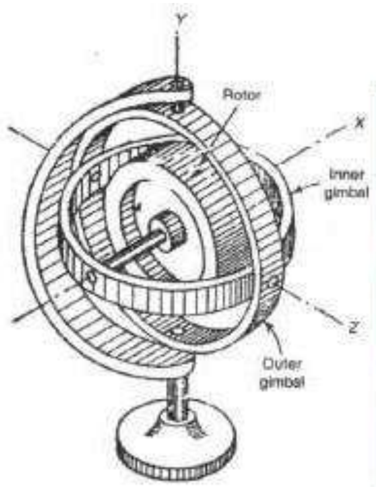
UNIT 2

GYROSCOPE

Introduction

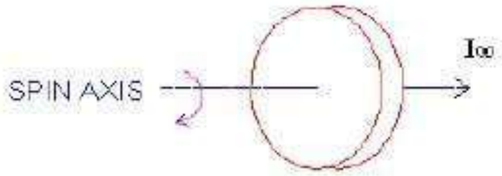
'Gyre' is a Greek word, meaning 'circular motion' and Gyration means the whirling motion. A gyroscope is a spatial mechanism which is generally employed for the study of precessional motion of a rotary body. Gyroscope finds applications in gyrocompass, used in aircraft, naval ship, control system of missiles and space shuttle. The gyroscopic effect is also felt on the automotive vehicles while negotiating aturn.

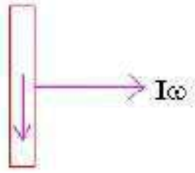
A gyroscope consists of a rotor mounted in the inner gimbal. The inner gimbal is mounted in the outer gimbal which itself is mounted on a fixed frame as shown in Fig.. When the rotor spins about X-axis with angular velocity ω rad/s and the inner gimbal precesses (rotates) about Y-axis, the spatial mechanism is forced to turn about Z-axis other than its own axis of rotation, and the gyroscopic effect is thus setup. The resistance to this motion is called gyroscopiceffect.



ANGULAR MOTION

A rigid body, (Fig.) spinning at a constant angular velocity ω rad/s about a spin axis through the mass centre. The angular momentum 'H' of the spinning body is represented by a **vector** whose magnitude is 'Iω'. I represents the mass amount of inertia of the rotor about the axis ofspin.

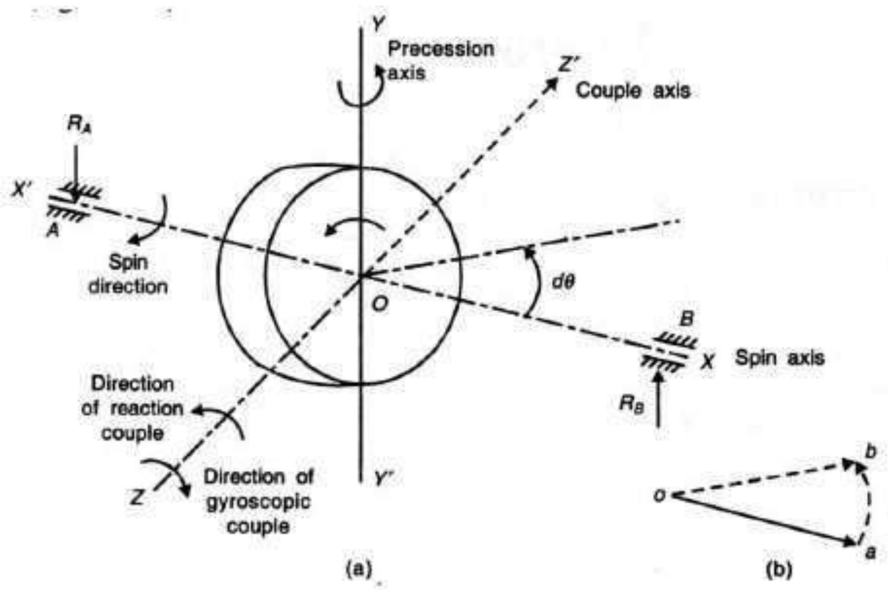




The direction of the angular momentum can be found from the right hand screw rule or the right hand thumb rule. Accordingly, if the fingers of the right hand are bent in the direction of rotation of rotor, then the thumb indicates the direction ofmomentum.

GYROSCOPIC COUPLE

Consider a rotary body of mass *m* having radius of gyration *k* mounted on the shaft supported at two bearings. Let the rotor spins (rotates) about X-axis with constant angular velocity ω rad/s. The X-axis is, therefore, called spin axis, Y-axis, precession axis and Z-axis, the couple or torque axis (Fig.).



The angular momentum of the rotating mass is given by,

$$H = mk^2 \omega = I\omega$$

Now, suppose the shaft axis (X-axis) precesses through a small angle $\delta\theta$ about Y-axis in the plane *XOZ,* then the angular momentum varies from *H* to *H* + δ*H,* where δ*H* is the change in the angular momentum, represented by vector ab [Figure 15.2(b)]. For the small value of angle of rotation 5o, we can write

$$ab = oa \times \delta\theta$$
$$\delta H = H \times \delta\theta$$
$$= I\omega\delta\theta$$

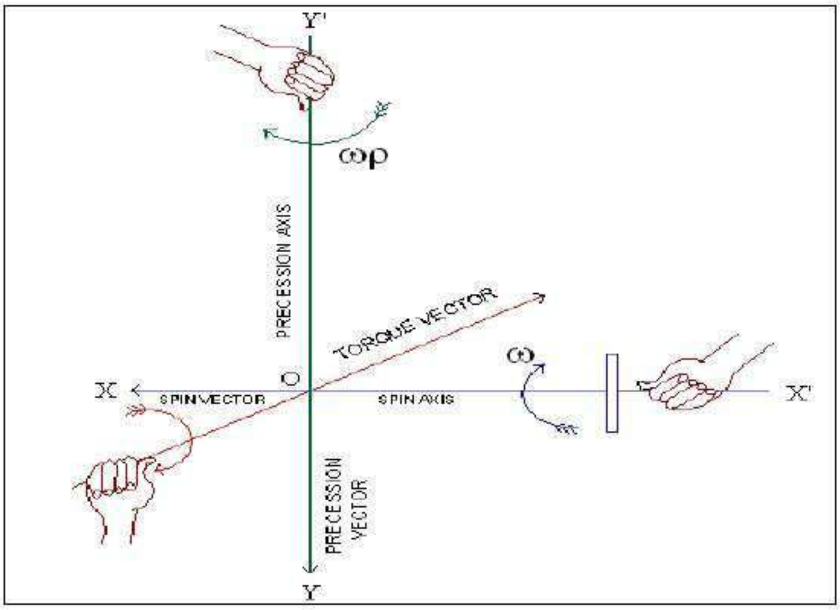
However, the rate of change of angular momentum is:

$$C = \frac{dH}{dt} = \lim_{\delta t \to 0}\left(\frac{I\omega\,\delta\theta}{\delta t}\right)$$
$$= I\omega\frac{d\theta}{dt}$$

$$\mathbf{C = I\omega\omega_p}$$

Direction of Spin vector, Precession vector and Couple/Torque vector with forced precession

To determine the direction of spin, precession and torque/couple vector, right hand screw rule or right hand rule is used. The fingers represent the rotation of the disc and the thumb



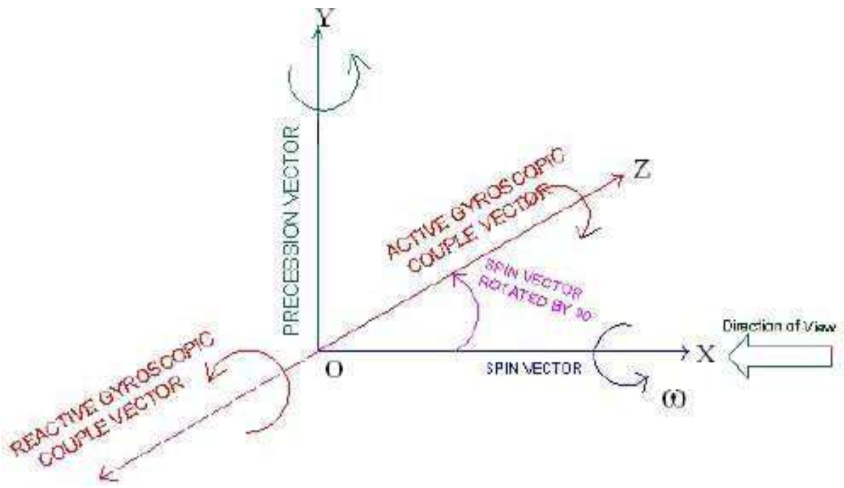
shows the direction of the spin, precession and torque vector (Fig.).

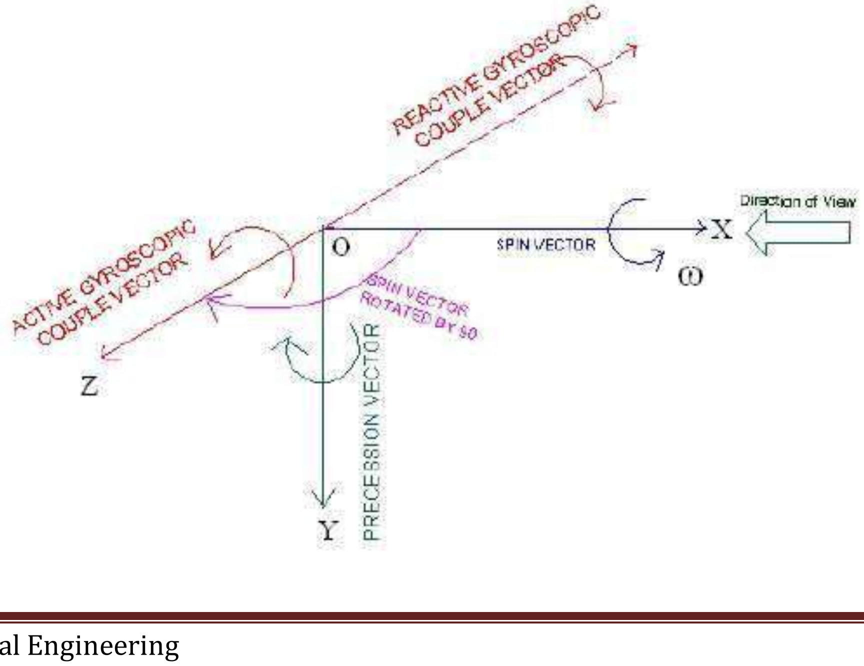
The method of determining the direction of couple/torque vector is as follows

Case (i):

Consider a rotor rotating in anticlockwise direction when seen from the right (Fig.5 and Fig. 6), and to precess the spin axis about precession axis in clockwise and anticlockwise direction when seen from top. Then, to determine the active/reactive gyroscopic couple vector, the following procedure isused.

- Turn the spin vector through 90_0 in the direction of precession on the XOZplane
- The turned spin vector will then correspond to the direction of active gyroscopic couple/torquevector
- The reactive gyroscopic couple/torque vector is taken opposite to active gyro vector direction

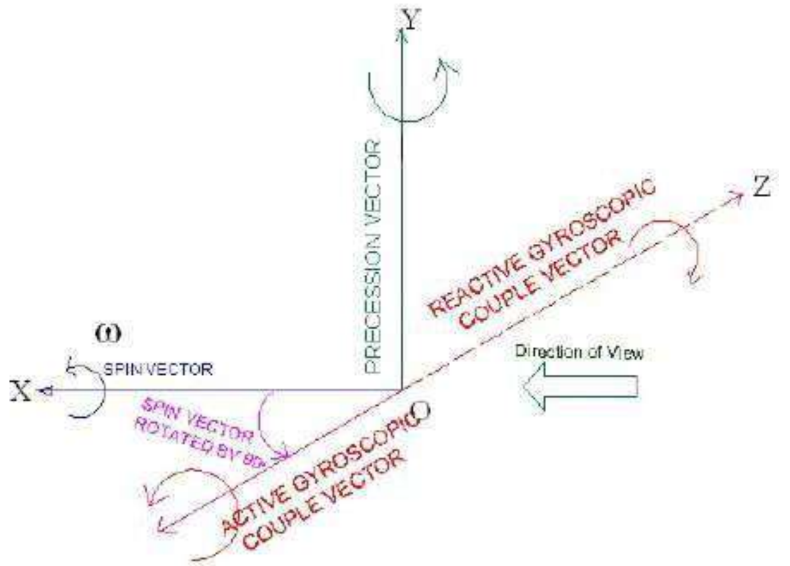


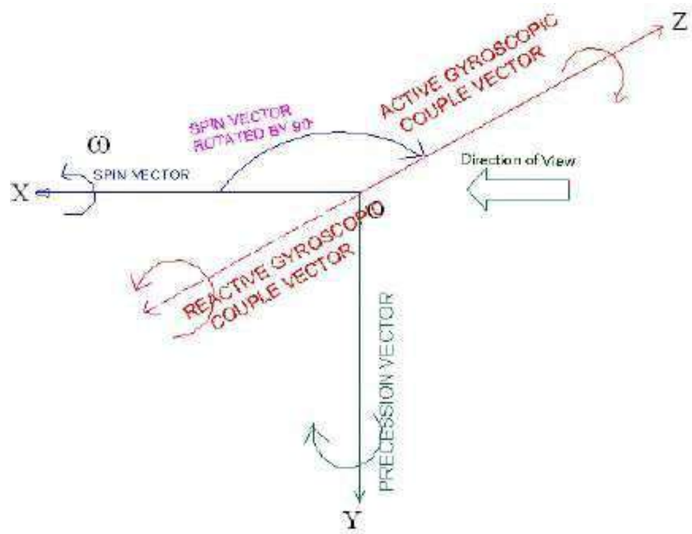


Case (ii):

Consider a rotor rotating in clockwise direction when seen from the right (Fig.7 and Fig. 8), and toprecess the spin axis about precession axis in clockwise and anticlockwise direction when seen from top. Then, to determine the active/reactive gyroscopic couplevector,

- Turn the spin vector through 90^0 in the direction of precession on the XOZplane
- The turned spin vector will then correspond to the direction of active gyroscopic couple/torquevector
- The reactive gyroscopic couple/torque vector is taken opposite to active gyro vector direction





The resisting couple/ reactive couple will act in the direction opposite to that of the gyroscopic couple. This means that, whenever the axis of spin changes its direction, a **gyroscopic couple** is applied to it through the bearing which supports the spinning axis.

GYROSCOPIC EFFECT ON SHIP

Gyroscope is used for stabilization and directional control of a ship sailing in the rough sea. A ship, while navigating in the rough sea, may experience the following three different types ofmotion:

(i) Steering—The turning of ship in a curve while movingforward

(ii) Pitching—The movement of the ship up and down from horizontal position in a vertical plane about transverseaxis

(iii) Rolling—Sideway motion of the ship about longitudinalaxis

For stabilization of a ship against any of the above motion, the major requirement is that the gyroscope shall be made to precess in such a way that reaction couple exerted by the rotor opposes the disturbing couple which may act on theframe.

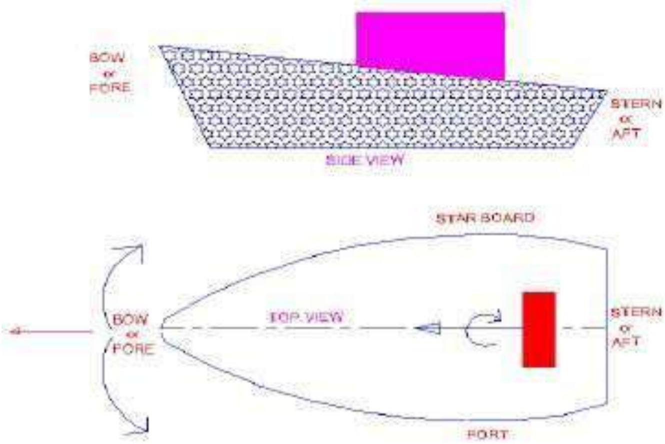
Ship Terminology

(i) Bow – It is the fore end ofship

(ii) Stern – It is the rear end ofship

(iii) Starboard – It is the right hand side of the ship looking in the direction ofmotion

(iv) Port – It is the left hand side of the ship looking in the direction ofmotion

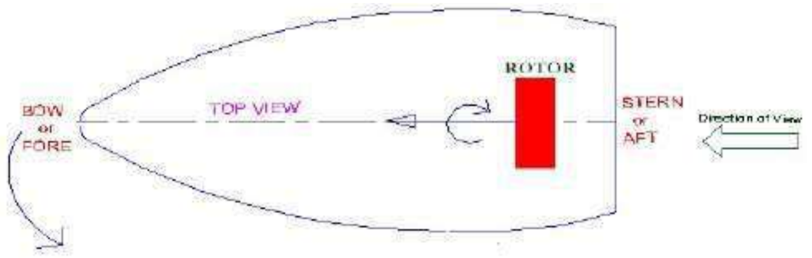


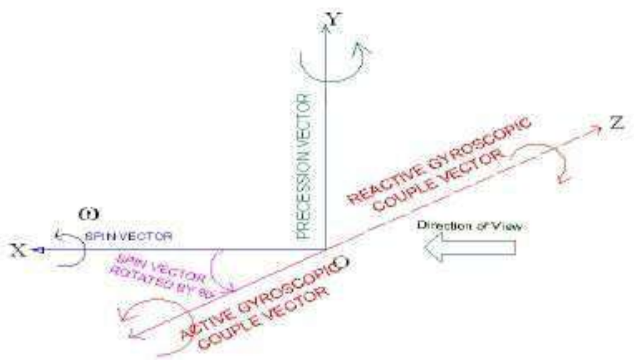
Consider a gyro-rotor mounted on the ship along longitudinal axis (X-axis) as shown in Fig.10 and rotate in clockwise direction when viewed from rear end of the ship. The angular speed of the rotor is ω rad/s. The direction of angular momentum vector *oa*, based on direction of rotation of rotor, is decided using right hand thumb rule as discussed earlier. The gyroscopic effect during the three types of motion of ship isdiscussed.

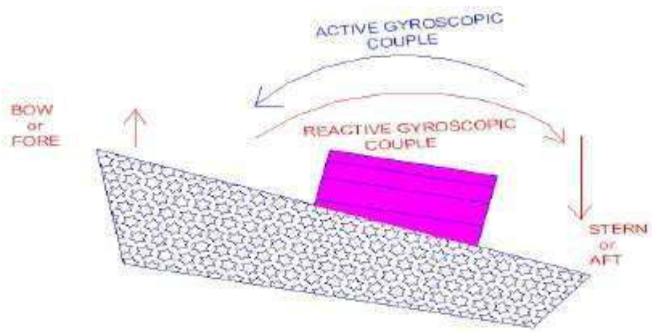
Gyroscopic effect on Steering of ship

(i) Left turn with clockwiserotor

When ship takes a left turn and the **rotor rotates in clockwise direction** viewed from stern, the gyroscopic couple act on the ship is analyzed in the followingway.



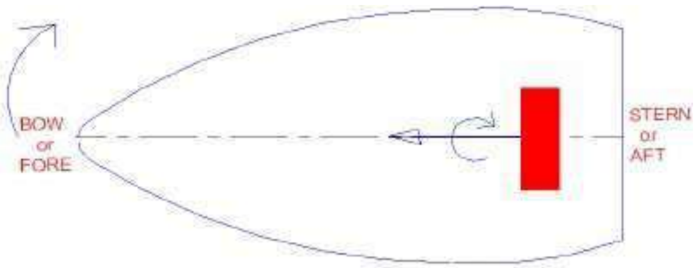


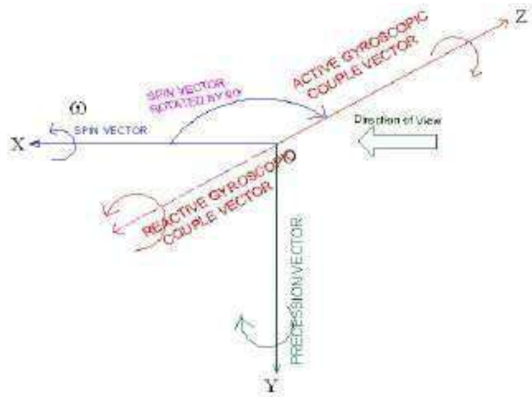


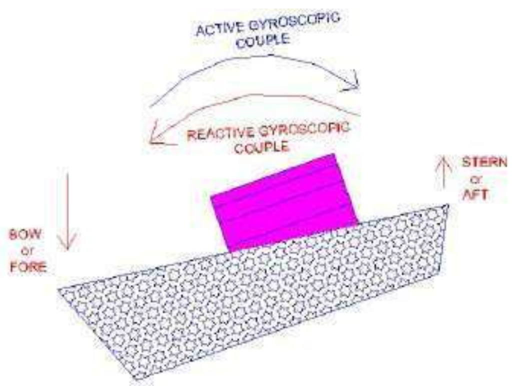
Note that, always reactive gyroscopic couple is considered for analysis. From the above analysis (Fig.), the couple acts over the ship between stern and bow. This reaction couple tends to raise the front end (bow) and lower the rear end (stern) of theship.

(ii) Right turn with clockwiserotor

When ship takes a right turn and the **rotor rotates in clockwise direction** viewed from stern, the gyroscopic couple acts on the ship is analyzed (Fig 14). Again, the couple acts in vertical plane, means between stern and bow. Now the reaction couple tends to lower the bow of the ship and raise thestern.

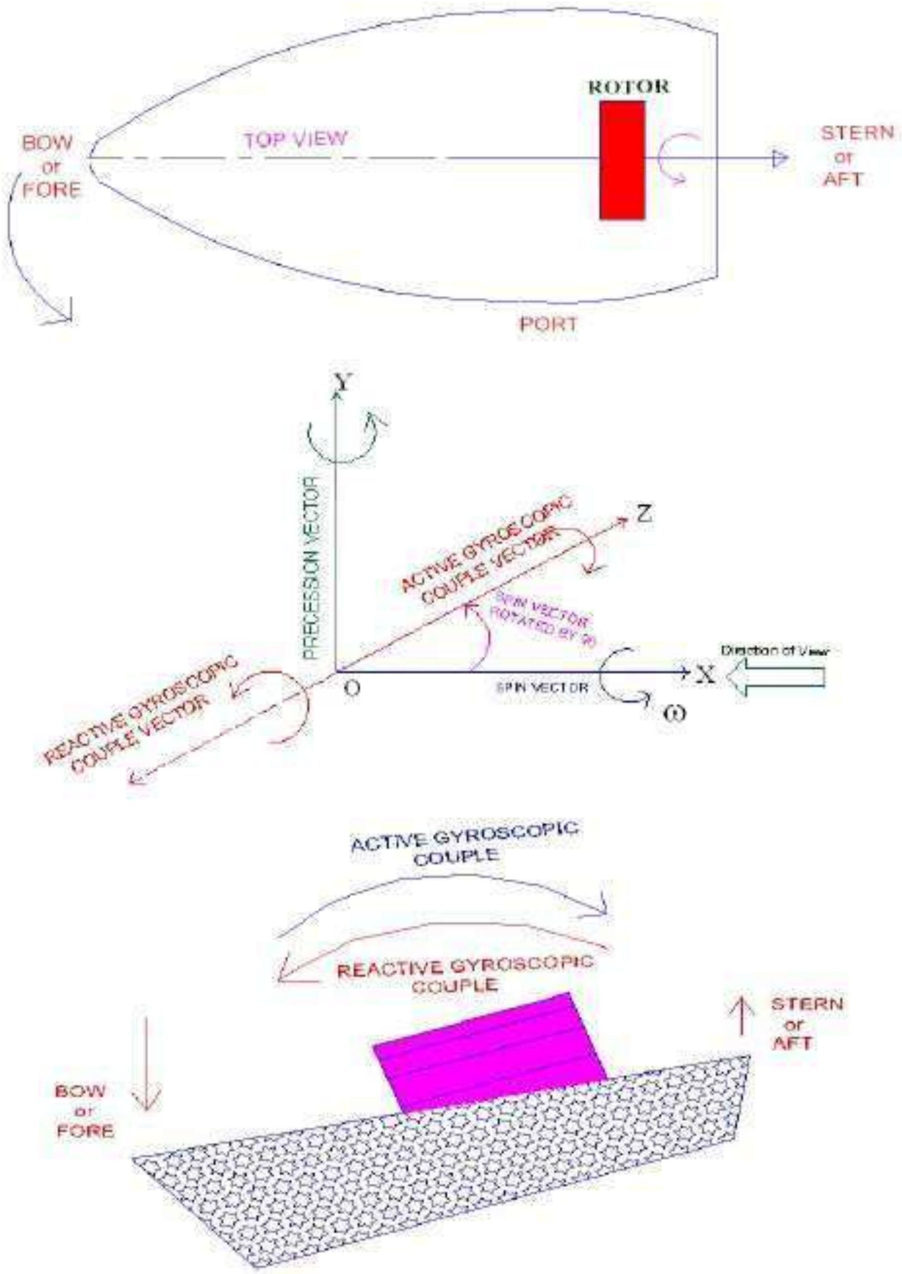






(iii) Left turn with anticlockwiserotor

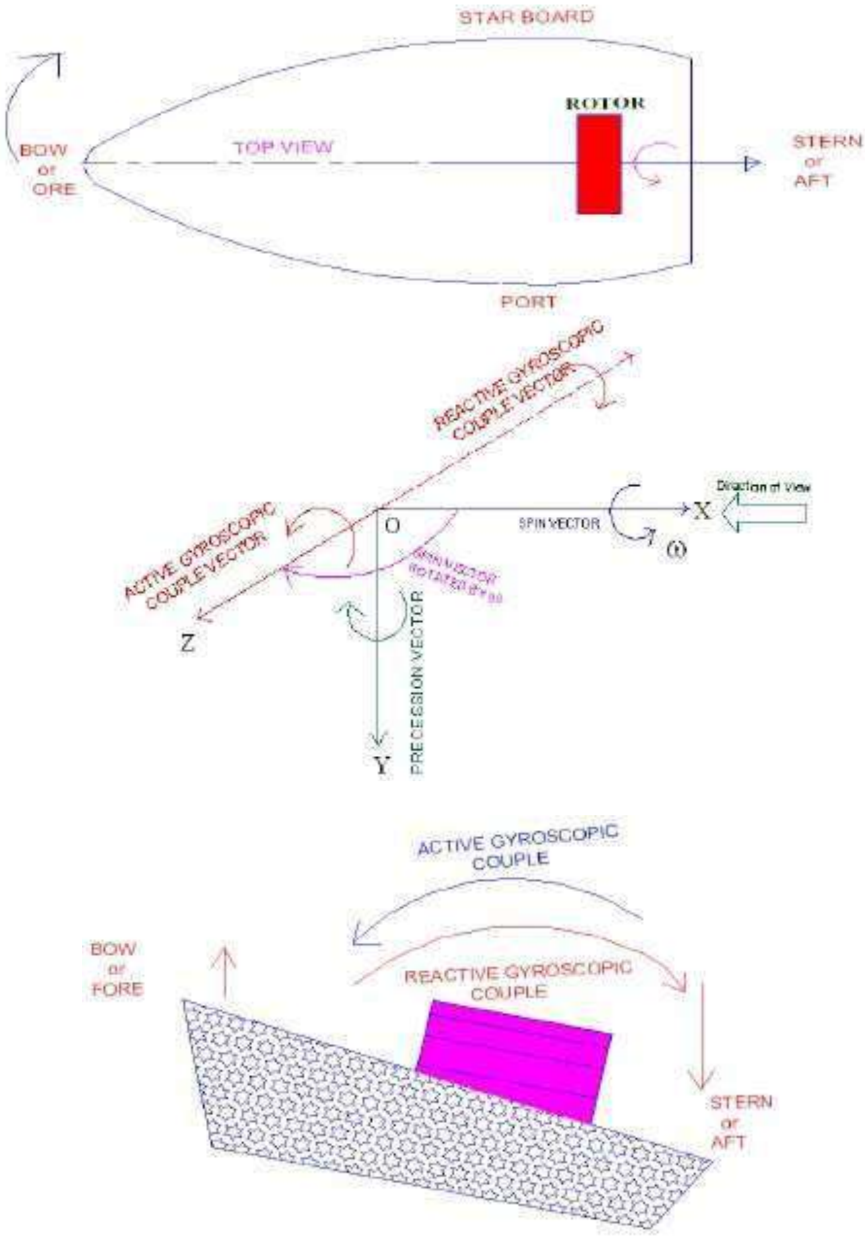
When ship takes a left turn and the **rotor rotates in anticlockwise direction** viewed from stern, the gyroscopic couple act on the ship is analyzed in the following way (Fig.).



The couple acts over the ship is between stern and bow. This reaction couple tends to press or dip the front end (bow) and raise the rear end (stern) of theship.

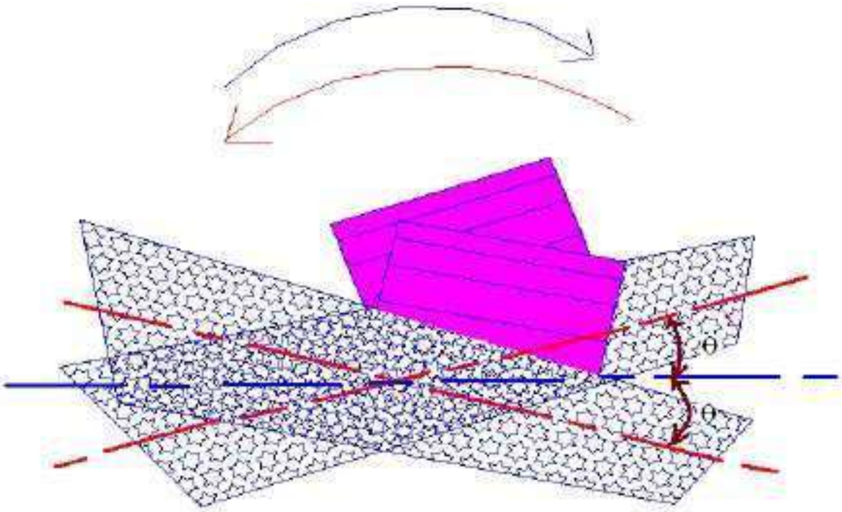
(iv) Right turn with anticlockwiserotor

When ship takes a right turn and the **rotor rotates in anticlockwise direction** viewed from stern, the gyroscopic couple act on the ship is according to Fig 20. Now, the reaction couple tends to raise the bow of the ship and dip thestern



Gyroscopic effect on Pitching of ship

The pitching motion of a ship generally occurs due to waves which can be approximated as sine wave. During pitching, the ship moves up and down from the horizontal position in vertical plane (Fig.)



Let θ = angular displacement of spin axis from its mean equilibrium position

A = amplitude of swing

(= angle in degree $\times \frac{2\pi}{360°}$)

and ω_0 = angular velocity of simple hormonic motion $\left(= \frac{2\pi}{\text{time period}}\right)$

The angular motion of the rotor is given as

$$\theta = A \sin \omega_0 t$$

Angular velocity of precess: $\omega_p = \frac{d\theta}{dt}$

$$= \frac{d}{dt}(A \sin \omega_0 t)$$

or $\omega_p = A\omega_0 \cos \omega_0 t$

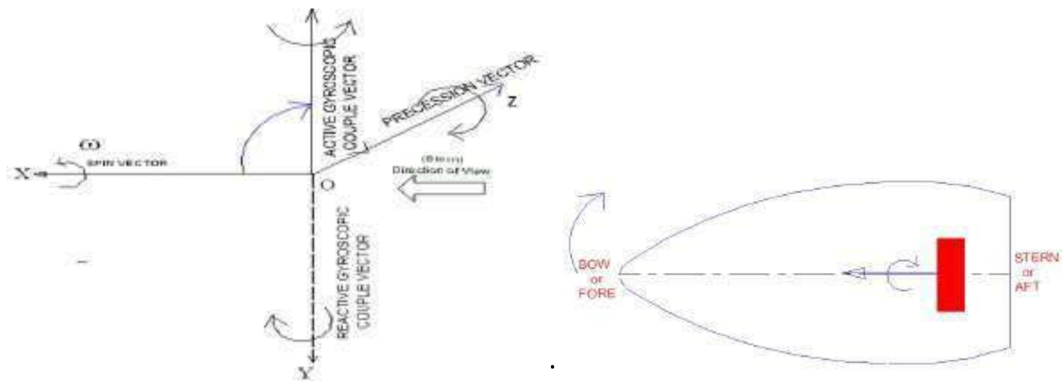
The angular velocity of precess will be maximum when $\cos \omega_0 t = 1$

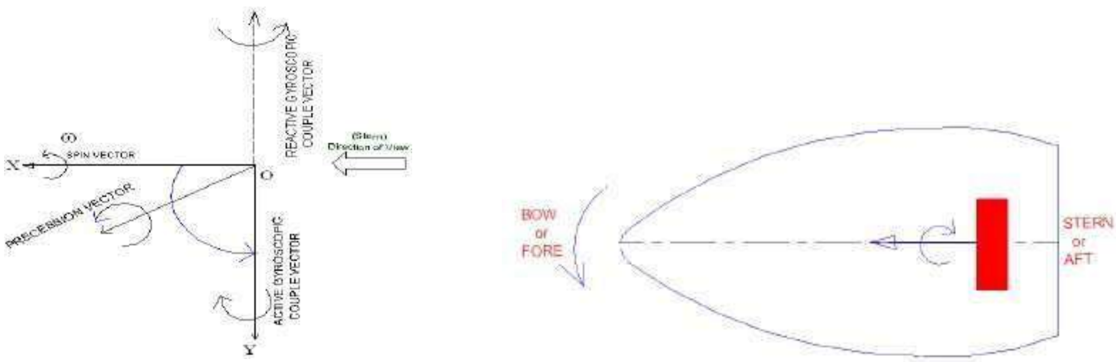
or $\omega_{p\max} = A\omega_0$

$$= A \times \frac{2\pi}{t}$$

Thus the gyroscopic couple: $C = I\,\omega\,\omega_p$

Consider a rotor mounted along the longitudinal axis and rotates in clockwise direction when seen from the rear end of the ship. The direction of momentum for this condition is shown by vector *ox* (Fig.24). When the ship moves up the horizontal position in vertical plane by an angle$\delta\theta$from the axis of spin, the rotor axis (X-axis) processes about Z- axis in XY-plane and for this case Z-axis becomes precession axis. The gyroscopic couple acts in anticlockwise direction about Y-axis and the reaction couple acts in opposite direction, i.e. in clockwise direction, which tends to move towards **right side** (Fig.25). However, when the ship pitches down the axis of spin, the direction of reaction couple is reversed and the ship turns towards **left side**(Fig.)





Similarly, for the anticlockwise direction of the rotor viewed from the rear end (Stern) of the ship, the analysis may be done.

Gyroscopic effect on Rolling of ship.

The axis of the rotor of a ship is mounted along the longitudinal axis of ship and therefore, there is **no** precession of this axis. Thus, no effect of gyroscopic couple on the ship frame is formed when the shiprolls

Gyroscopic Effect on Aeroplane

Aeroplanes are subjected to gyroscopic effect when it taking off, landing and negotiating left or right turn in the air.

Let

ω = Angular velocity of the engine rotating parts in rad/s,

m = Mass of the engine and propeller in kg,

r_W= Radius of gyration in m,

I = Mass moment of inertia of engine and propeller in kg m^2,

V = Linear velocity of the aeroplane in m/s,

R = Radius of curvature in m,

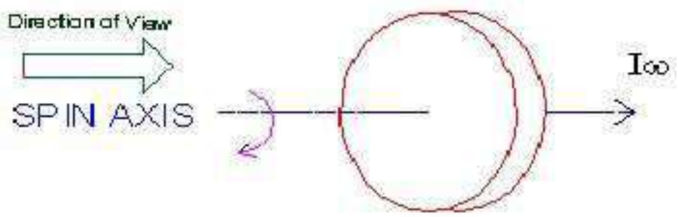
ω_p=Angular velocity of precession =v/R rad/s

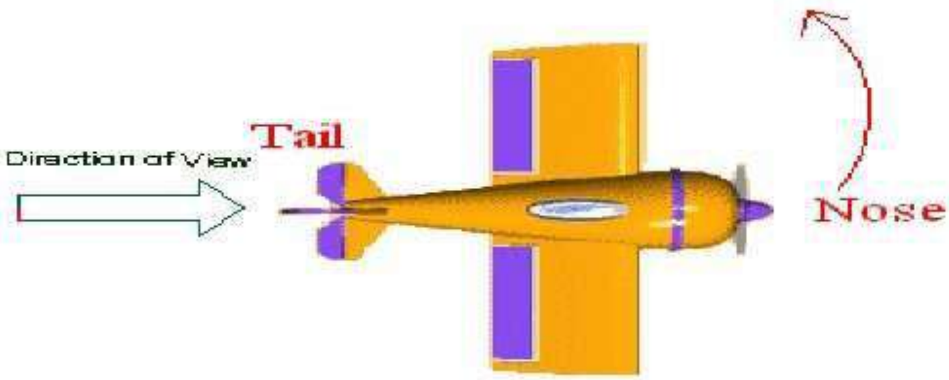
Gyroscopic couple acting on the aero plane **= C = I** $\omega\omega_p$

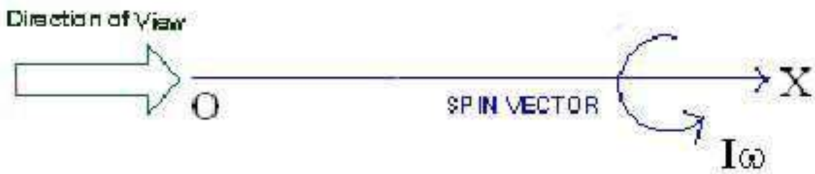
Let us analyze the effect of gyroscopic couple acting on the body of the aero plane for various conditions.

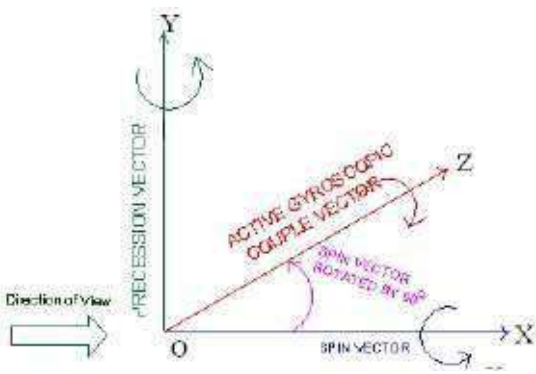
Case (i): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane turns towards LEFT

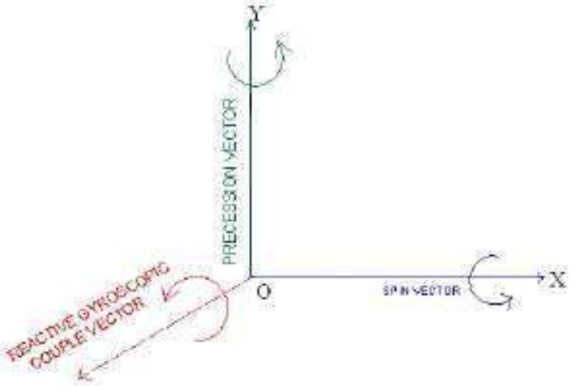

Tail
Direction of View
Nose


Direction of View
SPIN AXIS
Iω

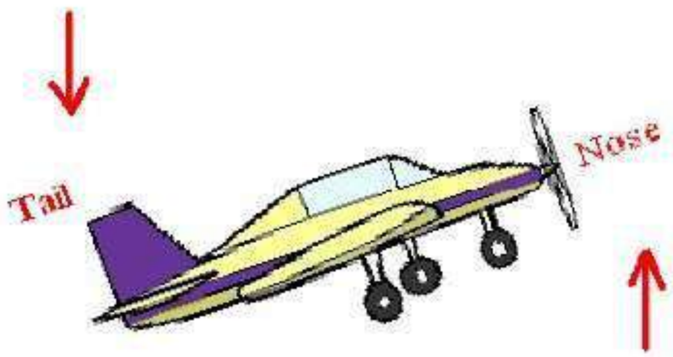

Direction of View
Tail
Nose


Direction of View
O
SPIN VECTOR
X
Iω

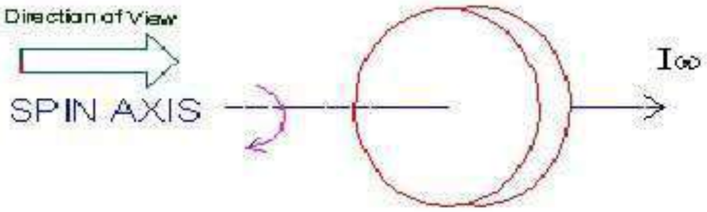

Y
PRECESSION VECTOR
ACTIVE GYROSCOPIC
COUPLE VECTOR
Z
Direction of View
O
SPIN VECTOR
X

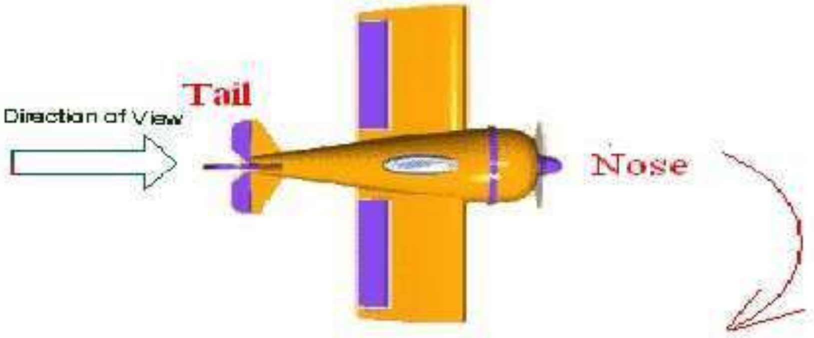


According to the analysis, the reactive gyroscopic couple tends to dip the tail and raise the nose ofaeroplane.

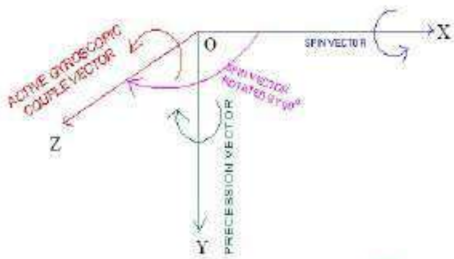


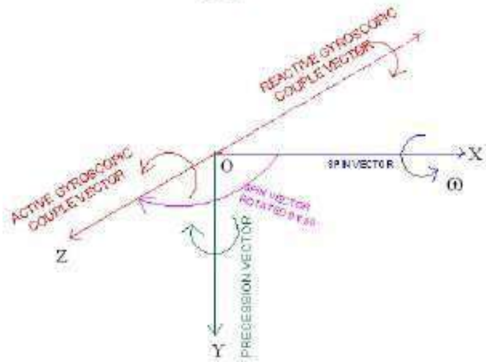
Case (ii): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane turns towardsRIGHT


Direction of View
SPIN AXIS
Iω


Tail
Direction of View
Nose

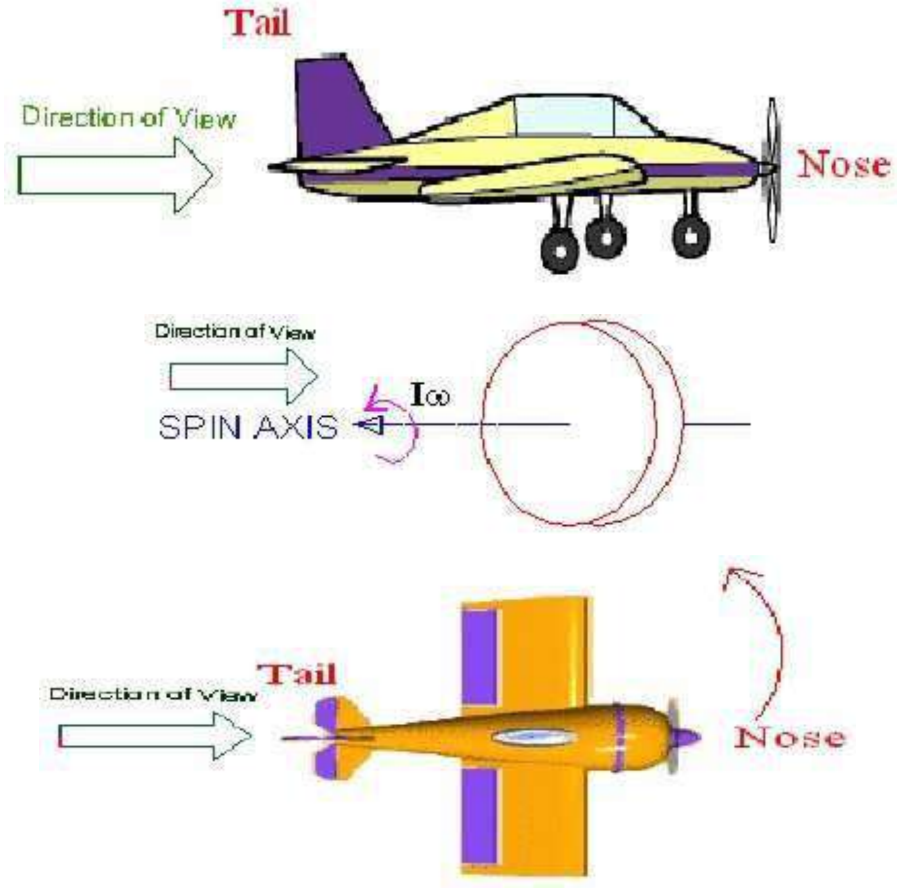

Direction of View
O
SPIN VECTOR
X
Iω

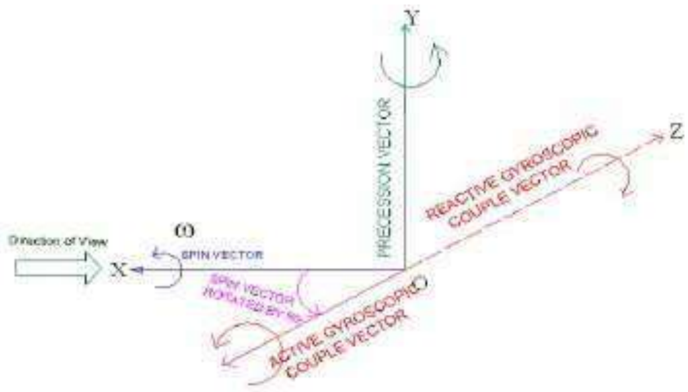

ACTIVE GYROSCOPIC COUPLE VECTOR
O
SPIN VECTOR
X
Z
PRECESSION VECTOR
Y


REACTIVE GYROSCOPIC COUPLE VECTOR
ACTIVE GYROSCOPIC COUPLE VECTOR
O
SPIN VECTOR
X
ω
Z
PRECESSION VECTOR
Y

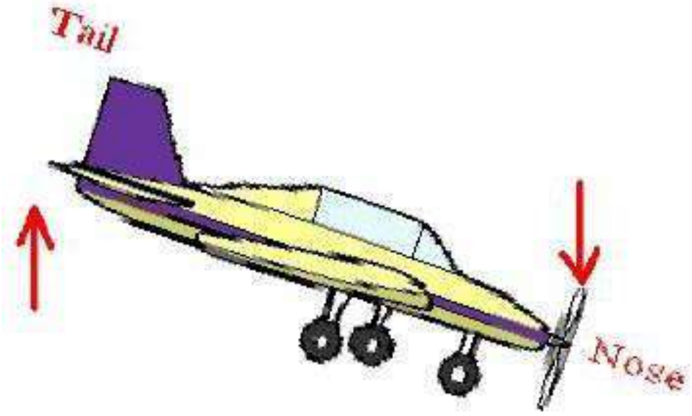
According to the analysis, the reactive gyroscopic couple tends to raise the tail and dip the nose ofaeroplane.

Case (iii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane turns towards LEFT

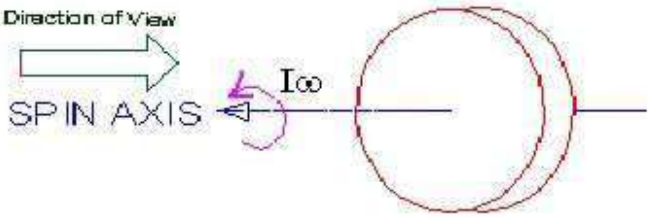


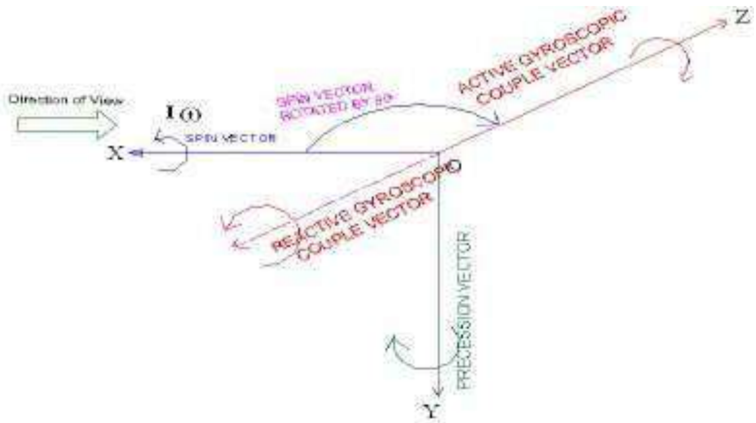


The analysis indicates, the reactive gyroscopic couple tends to raise the tail and dip the nose of aeroplane.



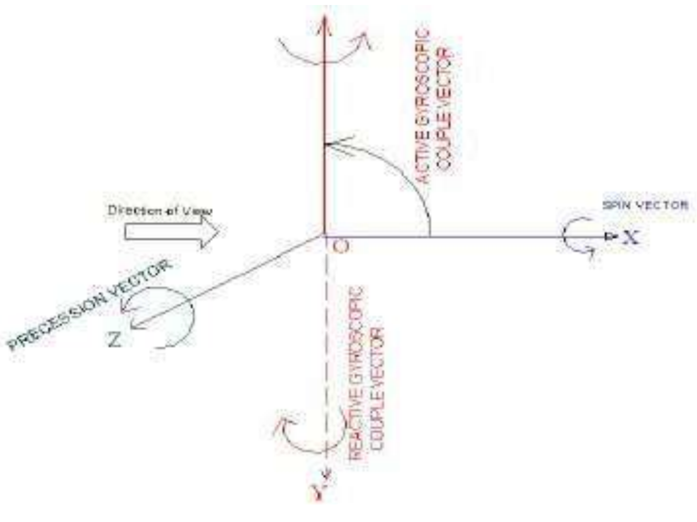
Case (iv): **PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane turns towards RIGHT**



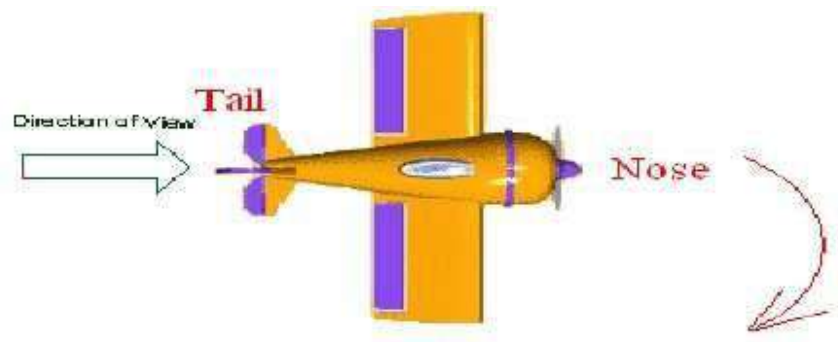


The analysis shows, the reactive gyroscopic couple tends to raise the tail and dip the nose of aeroplane.

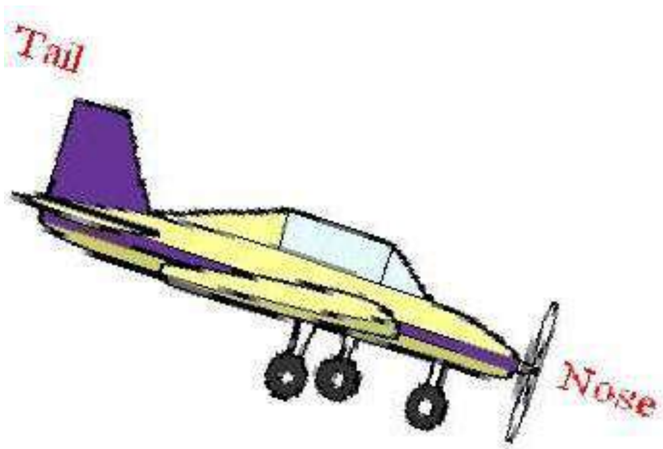
Case (v): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane takes off or nose moveupwards

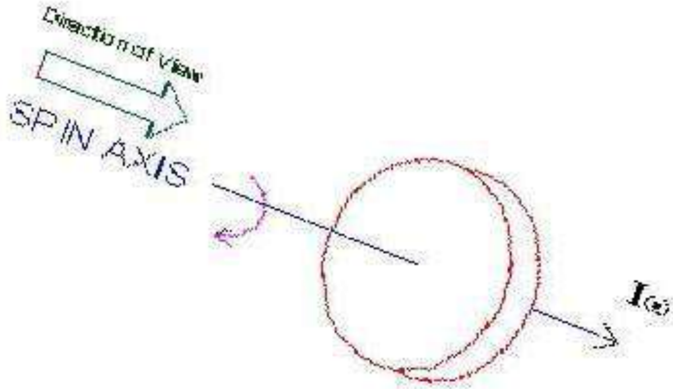


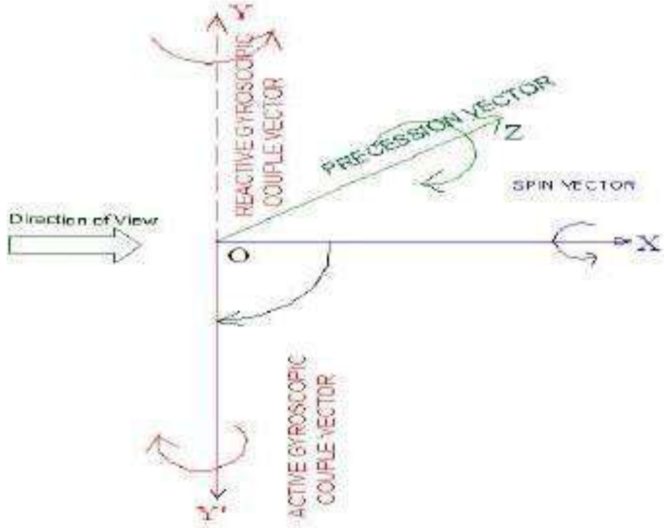
The analysis show, the reactive gyroscopic couple tends to turn the nose ofaeroplane toward right



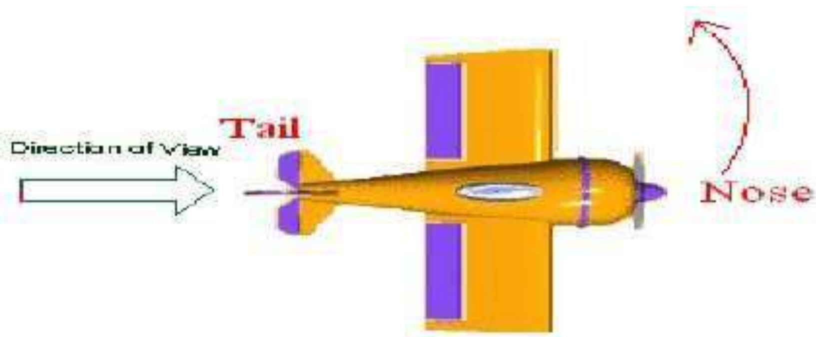
Case (vi): PROPELLER rotates in CLOCKWISE direction when seen from rear end and Aeroplane is landing or nose movedownwards



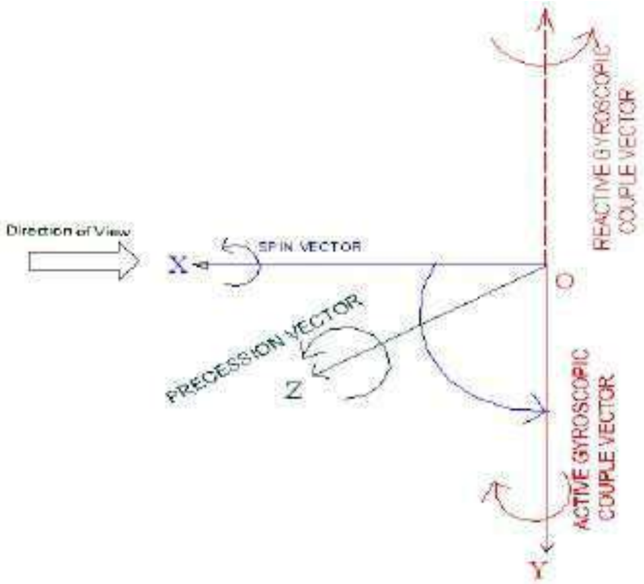




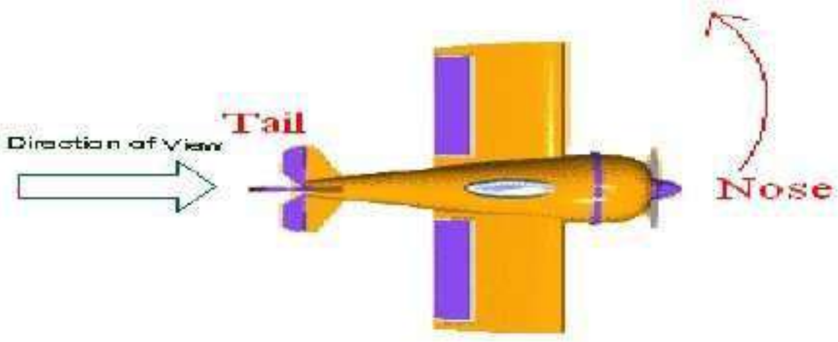
The reactive gyroscopic couple tends to turn the nose of aeroplane toward left



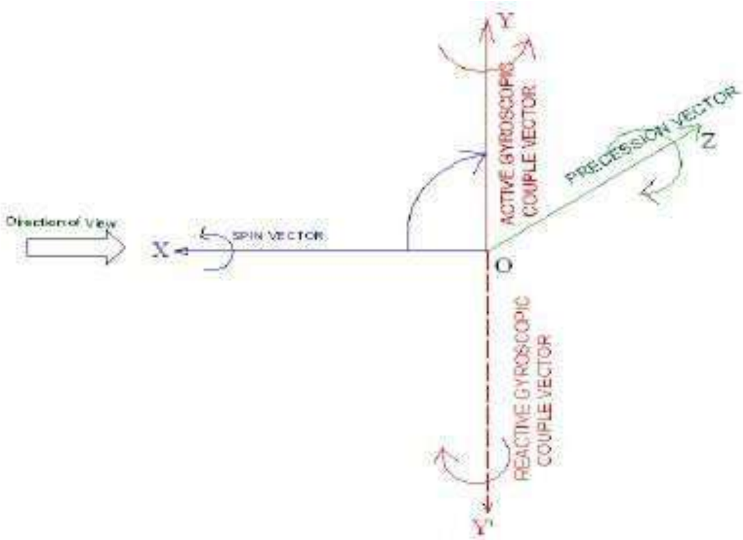
Case (vii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane takes off or nose move upwards



The reactive gyroscopic couple tends to turn the nose of aeroplane toward left



Case (viii): PROPELLER rotates in ANTICLOCKWISE direction when seen from rear end and Aeroplane is landing or nose move downwards



The analysis show, the reactive gyroscopic couple tends to turn the nose ofaeroplane toward right

Stability of Automotive Vehicle

A vehicle running on the road is said to be stable when no wheel is supposed to leave the road surface. In other words, the resultant reactions by the road surface on wheels should act in upward direction. For a moving vehicle, one of the reaction is due to gyroscopic couple produced by the rotating wheels and rotating parts of the engine. Let us discuss stability of two and four wheeled vehicles when negotiating a curve/turn.

Stability of Two Wheeler negotiating a turn

Fig shows a two wheeler vehicle taking **left turn** over a curved path. The vehicle is inclined to the vertical for equilibrium by an angle θ known as angle of heel.

Let

m = Mass of the vehicle and its rider in kg,

W = Weight of the vehicle and its rider in newtons = *m.g,*

h = Height of the centre of gravity of the vehicle and rider,

r_W *= Radius of thewheels,*

R = Radius of track or curvature,

I_W *= Mass moment of inertia of each wheel,*

I_E *= Mass moment of inertia of the rotating parts of the engine,*

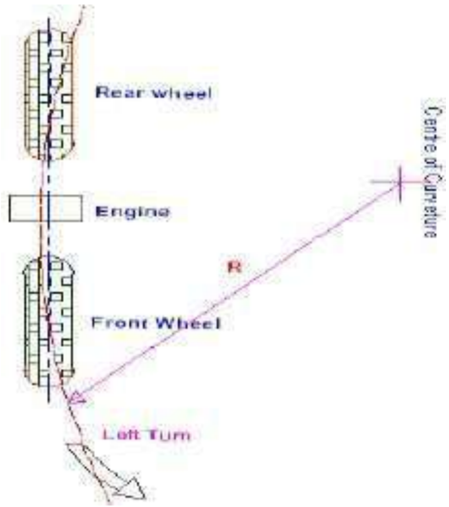
ω_W= Angular velocity of the wheels,

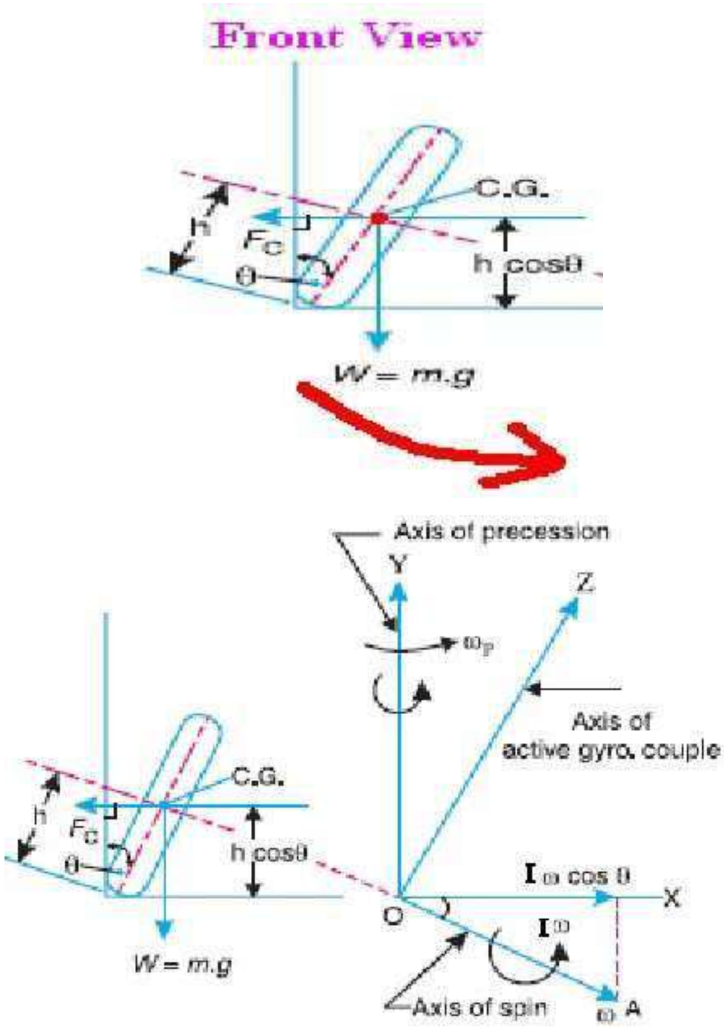
ω_E= Angular velocity of the engine rotating parts,

G = Gear ratio = ω_E / ω_W,

v = Linear velocity of the vehicle = $\omega_W \times r_W$,

θ = Angle of heel. It is inclination of the vehicle to the vertical for equilibrium





Let us consider the effect of the gyroscopic couple and centrifugal couple on the wheels.

1. Effect of GyroscopicCouple

Weknowthat, $V = \omega_W \times r_W$

$\omega_E = G\,.\omega_W$ or

Angular momentum due to wheels = 2 $I_w \omega_W$

Angular momentum due to engine and transmission = $I_E\,\omega_E$

Total angular momentum ($I\,x\omega$) = 2 $I_w\omega_W \pm I_E\omega_E$

$$= 2I_w\frac{v}{r_w} \pm I_E G\frac{v}{r_w}$$

$$= \frac{v}{r_w}(2I_w \pm GI_E)$$

Velocity of precession = ω_p

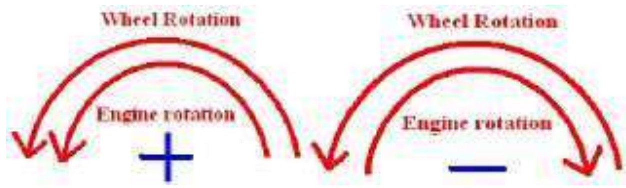
It is observed that, when the wheels move over the curved path, the vehicle is always inclined at an angle θ with the vertical plane as shown in Fig… This angle is known as 'angle of heel'. In other words, the axis of spin is inclined to the horizontal at an angle θ , as shown in Fig.73 Thus, the angular momentum vector I ω due to spin is represented by OA inclined to OX atanangleθ.But,theprecessionaxisisinvertical.Therefore,thespinvectorisresolvedalongOX.

Gyroscopic Couple,

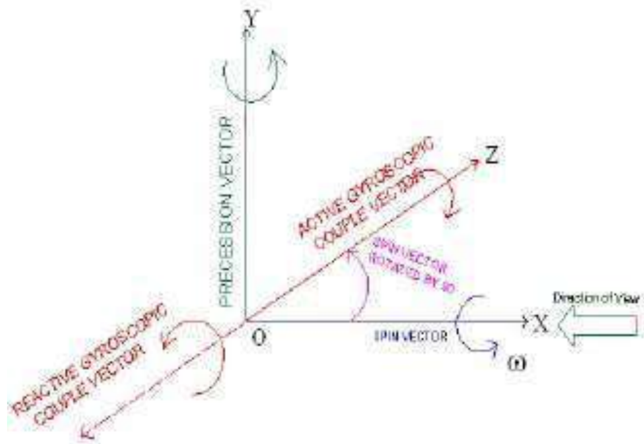
$$C_g = (I\omega)\cos\theta \times \omega_p$$

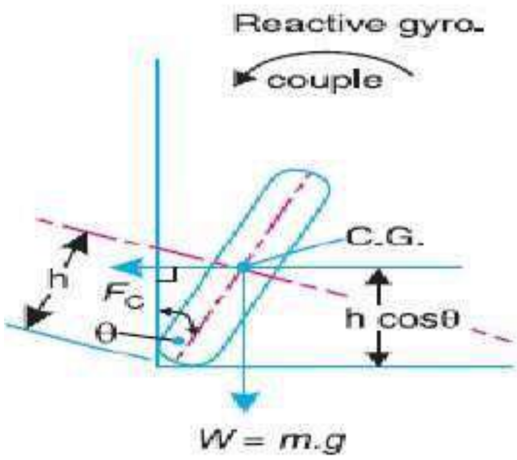
$$C_g = \frac{v^2}{Rr_w}(2I_w \pm GI_E)\cos\theta$$

Note: When the engine is rotating in the same direction as that of wheels, then the positive sign is used in the above equation. However, if the engine rotates in opposite direction to wheels, then negative sign isused.

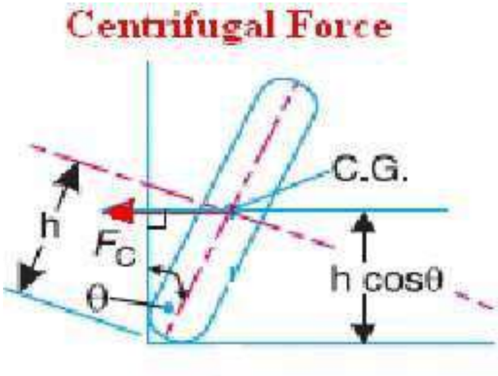


The gyroscopic couple will act over the vehicle outwards i.e., in the anticlockwise direction when seen from the front of the two wheeler. This couple tends to overturn/topple the vehicle in the outward direction as shown inFig…



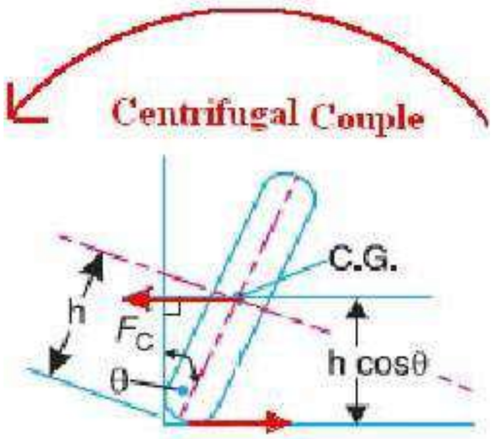


2. Effect of CentrifugalCouple



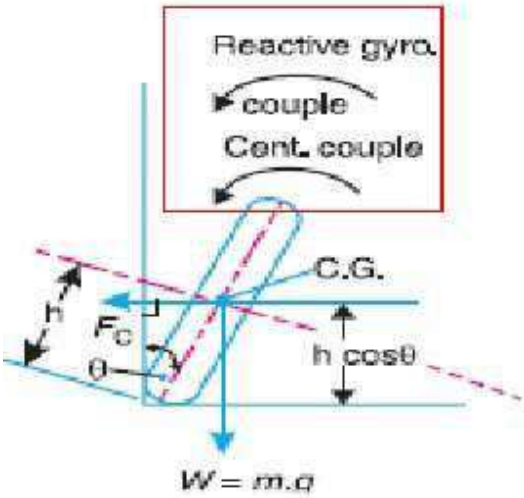
Centrifugal force, $$F_c = \frac{mv^2}{R}$$

Centrifugal Couple $$C_c = F_c \times h\cos\theta$$
$$= \frac{mv^2}{R} h\cos\theta$$



The Centrifugal couple will act over the two wheeler outwards i.e., in the anticlockwise direction when seen from the front of the two wheeler. This couple tends to overturn/topple the vehicle in the outward direction as shown in Fig.

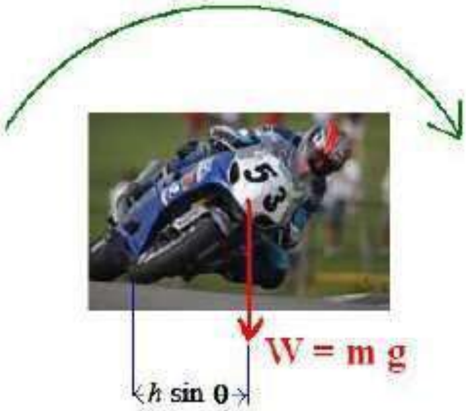
Therefore, the total Over turning couple: $C = C_g + C_c$



$$C = \frac{v^2}{Rr}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R}h\cos\theta$$

For the vehicle to be in equilibrium, overturning couple should be equal to balancing couple acting in clockwise direction due to the weight of the vehicle and rider.

$$C = mgh\sin\theta$$



For the stability, overturning couple must be equal to balancing couple,

$$\frac{v^2}{Rr_w}(2I_w + GI_e)\cos\theta + \frac{mv^2}{R}h\cos\theta = mgh\sin\theta$$

Therefore, from the above equation, the value of angle of heel (θ) may be determined, so that the vehicle does not skid. Also, for the given value of θ, the maximum vehicle speed in the turn with out skid may be determined.

Stability of Four Wheeled Vehicle negotiating a turn.

Consider a four wheels automotive vehicle as shown in Figure 82. The engine is mounted at the rear with its crank shaft parallel to the rear axle. The centre of gravity of the vehicle lies vertically above the ground where total weight of the vehicle is assumed to be acted upon.

Let

m = Mass of the vehicle (kg)

W = Weight of the vehicle (N) = *m.g,*

h = Height of the centre of gravity of the vehicle (m)

r_W= Radius of the wheels (m)

R = Radius of track or curvature (m)

I_W = Mass moment of inertia of each wheel (kg-m_2)

I_E = Mass moment of inertia of the rotating parts of the engine (kg-m_2)

ω_W= Angular velocity of the wheels (rad/s)

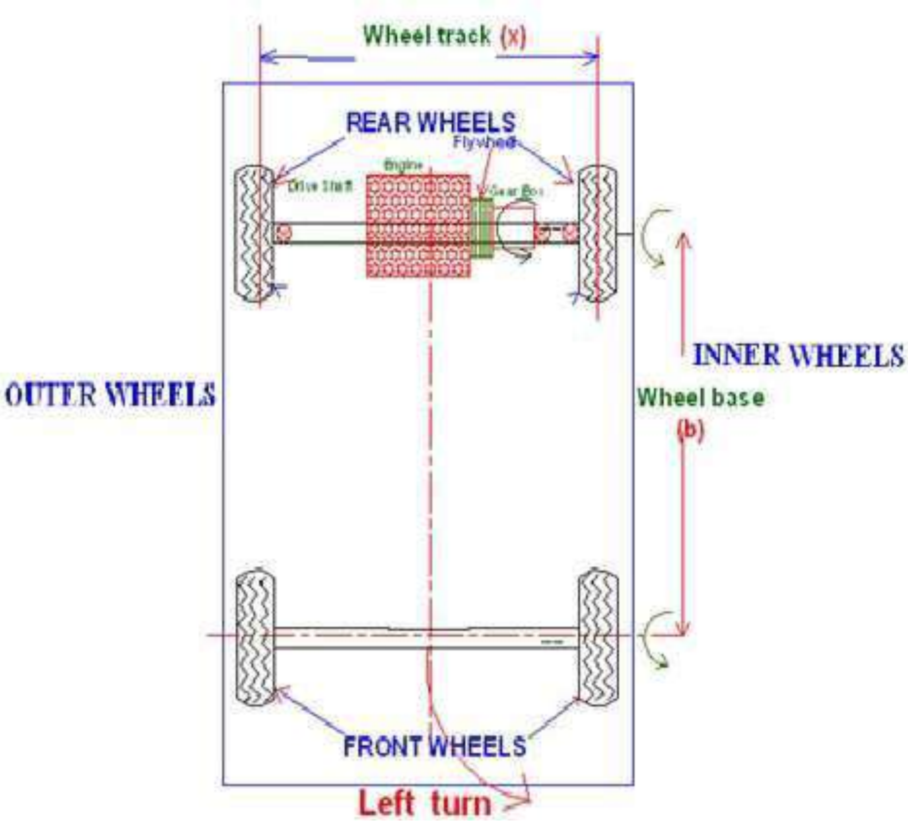
ω_E= Angular velocity of the engine (rad/s) *G*

= Gear ratio = ω_E / ω_W,

v = Linear velocity of the vehicle (m/s)=$\omega_W \times r_W$,

x = Wheel track (m)

b = Wheel base (m)



(i) Reaction due to weight ofVehicle

Weight of the vehicle.Assuming that weight of the vehicle *(mg)* is equally distributed over four wheels. Therefore, the force on each wheel acting downward is m*g/4* and the reaction by the road surface on the wheel acts in upward direction.

$$R_w = \frac{mg}{4}$$

(ii) Effect of Gyroscopic couple due toWheel

Gyroscopic couple due to four wheels is,

$$C_W = 4\, I_W \omega \omega_p$$

(iii) Effect of Gyroscopic Couple due toEngine

Gyroscopic couple due to rotating parts of the engine

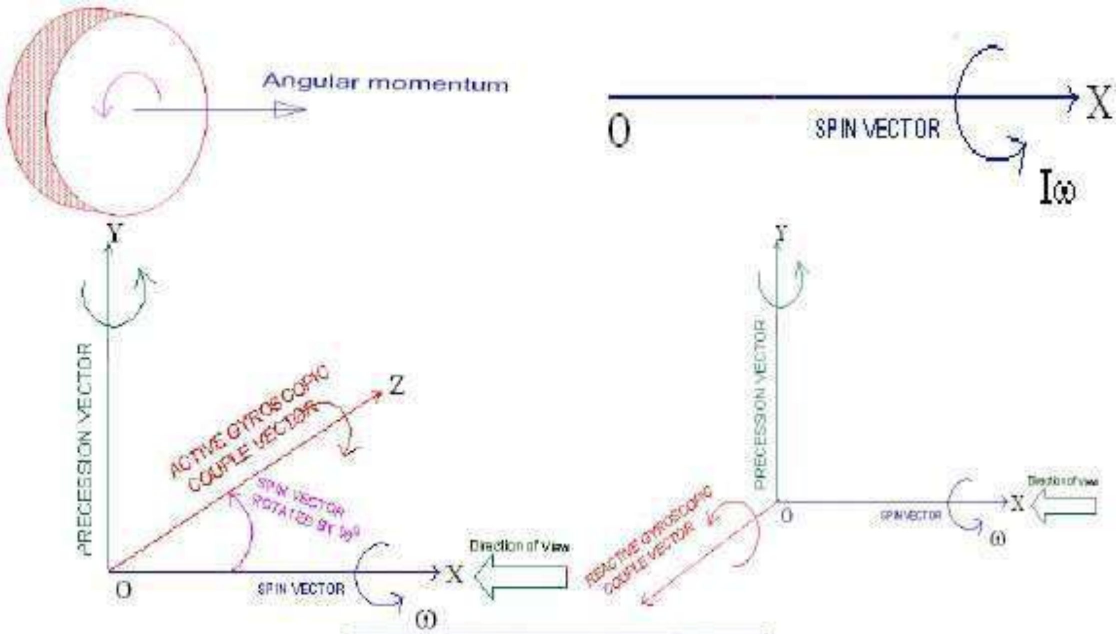
$$C_E = I_E\, \omega \omega_p = I_E\, G\, \omega \omega_p$$

Therefore, Total gyroscopic couple:

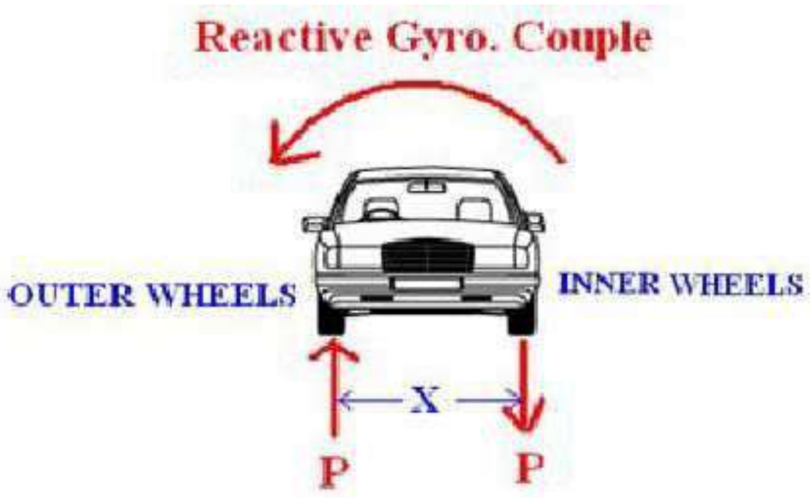
$$C_g = C_W + C_E = \omega\omega_p (4I_W \pm I_E G)$$

When the wheels and rotating parts of the engine rotate in the same direction, then positive sign is used in the above equation. Otherwise negative sign should beconsidered.

Assuming that the vehicle takes a left turn, the reaction gyroscopic couple on the vehicle acts between outer and inner wheels.



This gyroscopic couple tends to press the outer wheels and lift the inner wheels



Due to the reactive gyroscopic couple, vertical reactions on the road surface will be produced. The reaction will be vertically upwords on the outer wheels and vertically downwords on the inner wheels. Let the magnitude of this reaction at the two outer and inner wheels be P Newtons, then,

$$P \times X = C_g$$

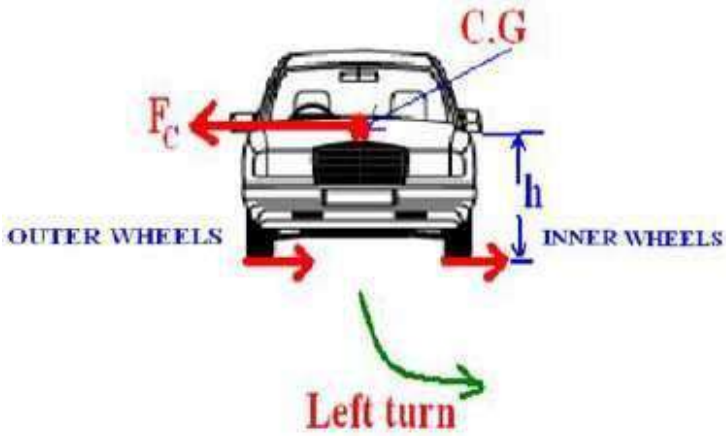
$$P = \frac{C_g}{X}$$

Road reaction on each outer/Inner wheel,

$$\frac{P}{2} = \frac{C_g}{2X}$$

(iii)**Effect of Centrifugal Couple**

When a vehicle moves on a curved path, a centrifugal force acts on the vehicle in outward direction through the centre of gravity of the vehicle(Fig…)


C.G
F_c
h
OUTER WHEELS
INNER WHEELS
Left turn

Centrifugal force,

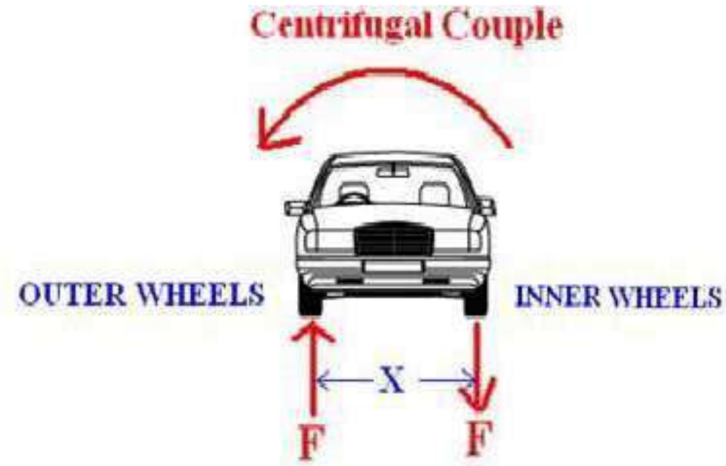
$$F_c = m\omega_p^2 R = \frac{mv^2}{R}$$

This force forms a Centrifugal couple.

$$C_c = \frac{mv^2 h}{R}$$

This centrifugal couple tends to press the outer and lift the inner

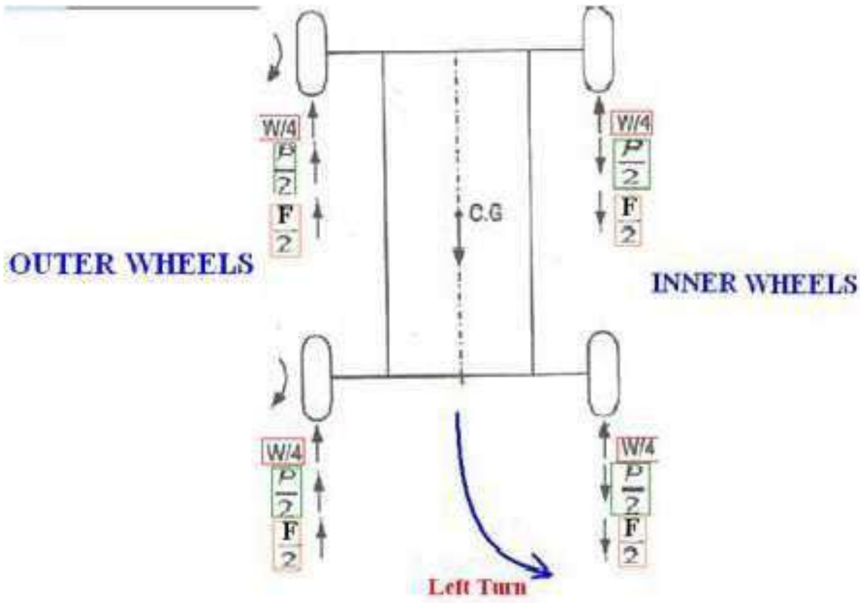
Due to the centrifugal couple, vertical reactions on the road surface will be produced. The reaction will be vertically upwords on the outer wheels and vertically downwords on the inner wheels. Let the magnitude of this reaction at the two outer and inner wheels be F Newtons, then,



Road reaction on each outer/Inner wheel,

$$\frac{F}{2} = \frac{C_c}{2X}$$

The reactions on the outer/inner wheels are as follows,



Total vertical reaction at each outerwheels

$$P_o = \frac{W}{4} + \frac{P}{2} + \frac{Q}{2}$$

Total vertical reaction at each innerwheels

$$P_i = \frac{W}{4} - \frac{P}{2} - \frac{Q}{2}$$

TURNING MOMENT DIAGRAM AND FLY WHEELS

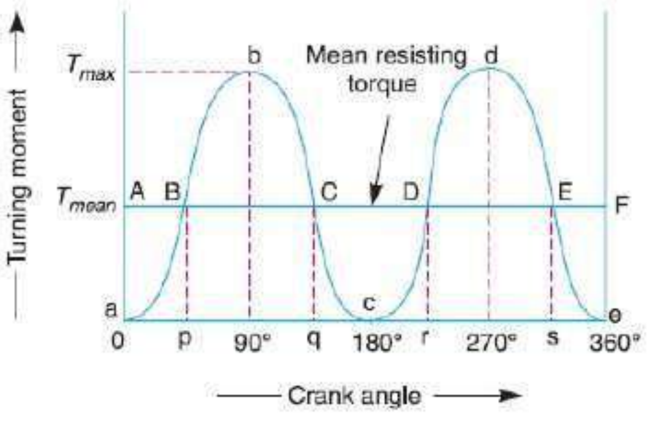
Turning Moment Diagram: The turning moment diagram is graphical representation of the turning moment or crank effort for various positions of crank.

Single cylinder double acting engine:

A turning moment diagram for a single cylinder double acting steam engine is shown in Fig. The vertical ordinate represents the turning moment and the horizontal ordinate represents the crank angle.

the turning moment on the crankshaft,

$$T = F_P \times r \left(\sin\theta + \frac{\sin 2\theta}{2\sqrt{n^2 - \sin^2\theta}} \right)$$



Turning moment diagram for a single cylinder, double acting steam engine.

where F_P = Piston effort,

r = Radius of crank,

n = Ratio of the connecting rod length and radius of crank, and

θ = Angle turned by the crank from inner dead centre.

From the above expression, we see that the turning moment (T) is zero, when the crank angle (θ) is zero. It is maximum when the crank angle is 90° and it is again zero when crank angle is 180°.

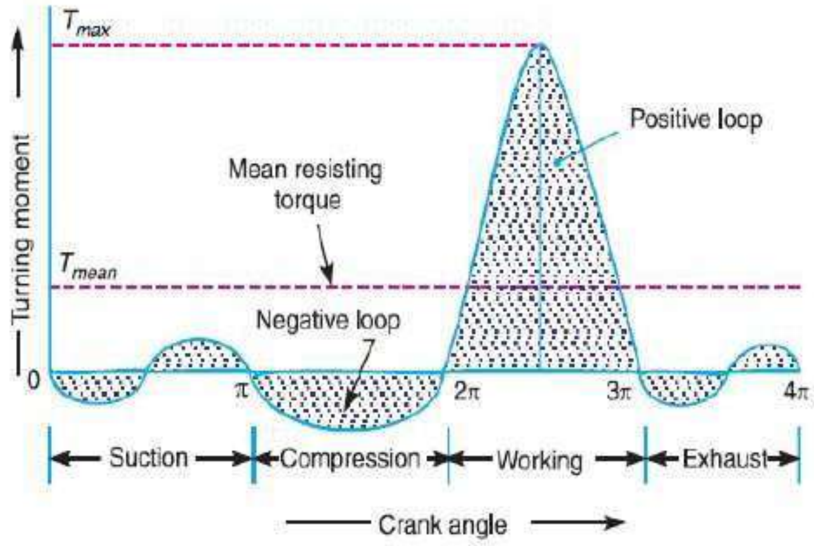
This is shown by the curve *abc* in Fig. and it represents the turning moment diagram for outstroke. The curve *cde* is the turning moment diagram for instroke and is somewhat similar to the curve *abc*.

Since the work done is the product of the turning moment and the angle turned, therefore the area of the turning moment diagram represents the work done per revolution. In actual practice, the engine is assumed to work against the mean resisting torque, as shown by a horizontal line *AF*. The height of the ordinate *aA* represents the mean height of the turning moment diagram. Since it is assumed that the work done by the turning moment per revolution is equal to the work done against the mean resisting torque, therefore the area of the rectangle *aAFe* is proportional to the work done against the mean resisting torque.

For flywheel, have a look at your tailor's manual sewing machine.

Turning moment diagram for 4-stroke I.C engine:

A turning moment diagram for a four stroke cycle internal combustion engine is shown in Fig. We know that in a four stroke cycle internal combustion engine, there is one working stroke after the crank has turned through two revolutions, *i.e.* 720° (or 4 π radians).

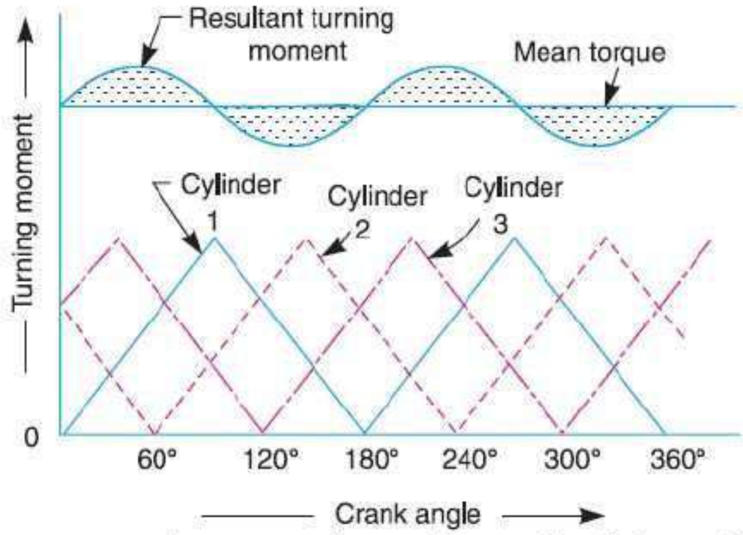


Turning moment diagram for a four stroke cycle internal combustion engine.

Since the pressure inside the engine cylinder is less than the atmospheric pressure during the suction stroke, therefore a negative loop is formed as shown in Fig. 16.2. During the compression stroke, the work is done on the gases, therefore a higher negative loop is obtained. During the expansion or working stroke, the fuel burns and the gases expand, therefore a large positive loop is obtained. In this stroke, the work is done by the gases. During exhaust stroke, the work is done on the gases, therefore a negative loop is formed. It may be noted that the effect of the inertia forces on the piston is taken into account in Fig.

Turning moment diagram for a multi cylinder engine:

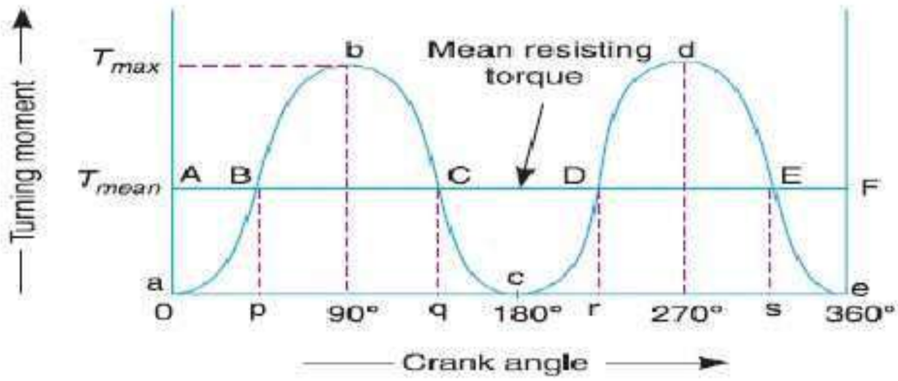
A separate turning moment diagram for a compound steam engine having three cylinders and the resultant turning moment diagram is shown in Fig. The resultant turning moment diagram is the sum of the turning moment diagrams for the three cylinders. It may be noted that the first cylinder is the high pressure cylinder, second cylinder is the intermediate cylinder and the third cylinder is the low pressure cylinder. The cranks, in case of three cylinders, are usually placed at 120° to each other.



Turning moment diagram for a multi-cylinder engine.

Fluctuation of Energy:

The difference in the kinetic energies at the point is called the maximum fluctuation of energy.



The fluctuation of energy may be determined by the turning moment diagram for one complete cycle of operation. Consider the turning moment diagram for a single cylinder double acting steam engine as shown in Fig. We see that the mean resisting torque line *AF* cuts the turning moment diagram at points *B*, *C*, *D* and *E*. When the crank moves from *a* to *p*, the work done by the engine is equal to the area *aBp*, whereas the energy required is represented by the area *aABp*. In other words, the engine has done less work (equal to the area *a AB*) than the requirement. This amount of energy is taken from the flywheel and hence the speed of the flywheel decreases. Now the crank moves from *p* to *q*, the work done by the engine is equal to the area *pBbCq*, whereas the requirement of energy is represented by the area *pBCq*. Therefore, the engine has done more work than the requirement. This excess work (equal to the area *BbC*) is stored in the flywheel and hence the speed of the flywheel increases while the crank moves from *p* to *q*.

Similarly, when the crank moves from *q* to *r*, more work is taken from the engine than is developed. This loss of work is represented by the area *C c D*. To supply this loss, the flywheel gives up some of its energy and thus the speed decreases while the crank moves from *q* to *r*. As the crank moves from *r* to *s*, excess energy is again developed given by the area *D d E* and the speed again increases. As the piston moves from *s* to *e*, again there is a loss of work and the speed decreases. The variations of energy above and below the mean resisting torque line are called ***fluctuations of energy***. The areas *BbC*, *CcD*, *DdE*, etc. represent fluctuations of energy.

A little consideration will show that the engine has a maximum speed either at *q* or at *s*. This is due to the fact that the flywheel absorbs energy while the crank moves from *p* to *q* and from *r* to *s*. On the other hand, the engine has a minimum speed either at *p* or at *r*. The reason is that the flywheel gives out some of its energy when the crank moves from *a* to *p* and *q* to *r*. The difference between the maximum and the minimum energies is known as ***maximum fluctuation of energy***.

Fluctuation of Speed:

This is defined as the ratio of the difference between the maximum and minimum angular speeds during a cycle to the mean speed of rotation of the crank shaft.

A turning moment diagram for a multi-cylinder engine is shown by a wavy curve in Fig. The horizontal line *A G* represents the mean torque line. Let a_1, a_3, a_5 be the areas above the mean torque line and a_2, a_4 and a_6 be the areas below the mean torque line. These areas represent some quantity of energy which is either added or subtracted from the energy of the moving parts of the engine.

Maximum fluctuation of energy:

Let the energy in the flywheel at $A = E$, then from Fig. we have

Energy at $B = E + a_1$

Energy at $C = E + a_1 - a_2$

Energy at $D = E + a_1 - a_2 + a_3$

Energy at $E = E + a_1 - a_2 + a_3 - a_4$

Energy at $F = E + a_1 - a_2 + a_3 - a_4 + a_5$

Energy at $G = E + a_1 - a_2 + a_3 - a_4 + a_5 - a_6$

= Energy at A (*i.e.* cycle repeats after G)

A flywheel stores energy when the supply is in excess and releases energy when energy is in deficit.

Let us now suppose that the greatest of these energies is at B and least at E. Therefore,

Maximum energy in flywheel

$$= E + a_1$$

Minimum energy in the flywheel

$$= E + a_1 - a_2 + a_3 - a_4$$

∴ Maximum fluctuation of energy,

$$\Delta E = \text{Maximum energy} - \text{Minimum energy}$$

$$= (E + a_1) - (E + a_1 - a_2 + a_3 - a_4) = a_2 - a_3 + a_4$$

<u>Coefficient of fluctuation of energy:</u>

It may be defined as the **ratio of the maximum fluctuation of energy to the work done per cycle.** Mathematically, coefficient of fluctuation of energy,

$$C_E = \frac{\text{Maximum fluctuation of energy}}{\text{Work done per cycle}}$$

The work done per cycle (in N-m or joules) may be obtained by using the following two relations :

1. Work done per cycle $= T_{mean} \times \theta$

where T_{mean} = Mean torque, and

θ = Angle turned (in radians), in one revolution.

= 2π, in case of steam engine and two stroke internal combustion engines

= 4π, in case of four stroke internal combustion engines.

The mean torque (T_{mean}) in N-m may be obtained by using the following relation :

$$T_{mean} = \frac{P \times 60}{2\pi N} = \frac{P}{\omega}$$

where P = Power transmitted in watts,

N = Speed in r.p.m., and

ω = Angular speed in rad/s $= 2\pi N/60$

2. The work done per cycle may also be obtained by using the following relation :

$$\text{Work done per cycle} = \frac{P \times 60}{n}$$

where n = Number of working strokes per minute,

= N, in case of steam engines and two stroke internal combustion engines,

= $N/2$, in case of four stroke internal combustion engines.

Coefficient of fluctuation of speed:

The difference between the maximum and minimum speeds during a cycle is called the ***maximum fluctuation of speed***. The ratio of the maximum fluctuation of speed to the mean speed is called the ***coefficient of fluctuation of speed***.

Let N_1 and N_2 = Maximum and minimum speeds in r.p.m. during the cycle, and

$$N = \text{Mean speed in r.p.m.} = \frac{N_1 + N_2}{2}$$

∴ Coefficient of fluctuation of speed,

$$C_S = \frac{N_1 - N_2}{N} = \frac{2(N_1 - N_2)}{N_1 + N_2}$$

$$= \frac{\omega_1 - \omega_2}{\omega} = \frac{2(\omega_1 - \omega_2)}{\omega_1 + \omega_2} \quad \text{...(In terms of angular speeds)}$$

$$= \frac{v_1 - v_2}{v} = \frac{2(v_1 - v_2)}{v_1 + v_2} \quad \text{...(In terms of linear speeds)}$$

Energy stored in flywheel:

A flywheel is a rotating mass that is used as an energy reservoir in a machine. It absorbs energy in the form of kinetic energy, during those periods of crank rotation when actual turning moment is greater than the resisting moment and release energy, by way of parting with some of its K.E, when the actual turning moment is less than the resisting moment

PROBLEMS

1 *The mass of flywheel of an engine is 6.5 tonnes and the radius of gyration is 1.8 metres. It is found from the turning moment diagram that the fluctuation of energy is 56 kN-m. If the mean speed of the engine is 120 r.p.m., find the maximum and minimum speeds.*

Solution. Given : m = 6.5 t = 6500 kg ; k = 1.8 m ; ΔE = 56 kN-m = 56 × 10^3 N-m ; N = 120 r.p.m.

Let N_1 and N_2 = Maximum and minimum speeds respectively.

We know that fluctuation of energy (ΔE),

$$56 \times 10^3 = \frac{\pi^2}{900} \times m.k^2 . N (N_1 - N_2) = \frac{\pi^2}{900} \times 6500 (1.8)^2 \, 120 (N_1 - N_2)$$

$$= 27\,715 (N_1 - N_2)$$

$$\therefore \quad N_1 - N_2 = 56 \times 10^3 / 27\,715 = 2 \text{ r.p.m.} \quad ...(i)$$

We also know that mean speed (N),

$$120 = \frac{N_1 + N_2}{2} \text{ or } N_1 + N_2 = 120 \times 2 = 240 \text{ r.p.m.} \quad ...(ii)$$

From equations (*i*) and (*ii*),

N_1 = 121 r.p.m., and N_2 = 119 r.p.m. **Ans.**

2. *The flywheel of a steam engine has a radius of gyration of 1 m and mass 2500 kg. The starting torque of the steam engine is 1500 N-m and may be assumed constant. Determine:* ***1.*** *the angular acceleration of the fly*

Solution. Given : k = 1 m ; m = 2500 kg ; T = 1500 N-m

1. Angular acceleration of the flywheel

Let α = Angular acceleration of the flywheel.

We know that mass moment of inertia of the flywheel,

$$I = m.k^2 = 2500 \times 1^2 = 2500 \text{ kg-m}^2$$

∴ Starting torque of the engine (T),

$$1500 = I.\alpha = 2500 \times \alpha \quad \text{or} \quad \alpha = 1500 / 2500 = 0.6 \text{ rad/s}^2$$ **Ans.**

2. Kinetic energy of the flywheel

First of all, let us find out the angular speed of the flywheel after 10 seconds from the start (*i.e.* from rest), assuming uniform acceleration.

Let ω_1 = Angular speed at rest = 0

ω_2 = Angular speed after 10 seconds, and

t = Time in seconds.

We know that $\omega_2 = \omega_1 + \alpha t = 0 + 0.6 \times 10 = 6$ rad/s

wheel, and ***2.*** *the kinetic energy of the flywheel after 10 seconds from the start.*

∴ Kinetic energy of the flywheel

$$= \frac{1}{2} \times I(\omega_2)^2 = \frac{1}{2} \times 2500 \times 6^2 = 45\,000 \text{ N-m} = 45 \text{ kN-m} \quad \textbf{Ans.}$$

3. A horizontal cross compound steam engine develops 300 kW at 90 r.p.m. The coefficient of fluctuation of energy as found from the turning moment diagram is to be 0.1 and the fluctuation of speed is to be kept within ± 0.5% of the mean speed. Find the weight of the flywheel required, if the radius of gyration is 2 metres.

Solution. Given : $P = 300$ kW $= 300 \times 10^3$ W; $N = 90$ r.p.m.; $C_E = 0.1$; $k = 2$ m

We know that the mean angular speed,

$$\omega = 2\pi N/60 = 2\pi \times 90/60 = 9.426 \text{ rad/s}$$

Let ω_1 and ω_2 = Maximum and minimum speeds respectively.

Since the fluctuation of speed is ± 0.5% of mean speed, therefore total fluctuation of speed,

$$\omega_1 - \omega_2 = 1\%\ \omega = 0.01\ \omega$$

nd coefficient of fluctuation of speed,

$$C_S = \frac{\omega_1 - \omega_2}{\omega} = 0.01$$

We know that work done per cycle

$$= P \times 60 / N = 300 \times 10^3 \times 60 / 90 = 200 \times 10^3 \text{ N-m}$$

∴ Maximum fluctuation of energy,

$$\Delta E = \text{Work done per cycle} \times C_E = 200 \times 10^3 \times 0.1 = 20 \times 10^3 \text{ N-m}$$

Let m = Mass of the flywheel.

We know that maximum fluctuation of energy (ΔE),

$$20 \times 10^3 = m.k^2.\omega^2.C_S = m \times 2^2 \times (9.426)^2 \times 0.01 = 3.554\ m$$

$$\therefore \quad m = 20 \times 10^3/3.554 = 5630 \text{ kg} \quad \textbf{Ans.}$$

UNIT 3

Governor

A governor, or speed limiter or controller, is a device used to measure and regulate the speed of a machine

Functions of Governor (2M Important)

- The function of a governor is to regulate the mean speed of an engine, when there are variations in the load.
- When the load on an engine increases, its speed decreases, therefore it becomes necessary to increases the supply of working fluid.
- When the load on the engine decreases, its speed increases and thus less working fluid is required.
- The governor automatically controls the supply of working fluid to the engine with the varying load conditions and keeps the mean speed of the engine within certain limits.

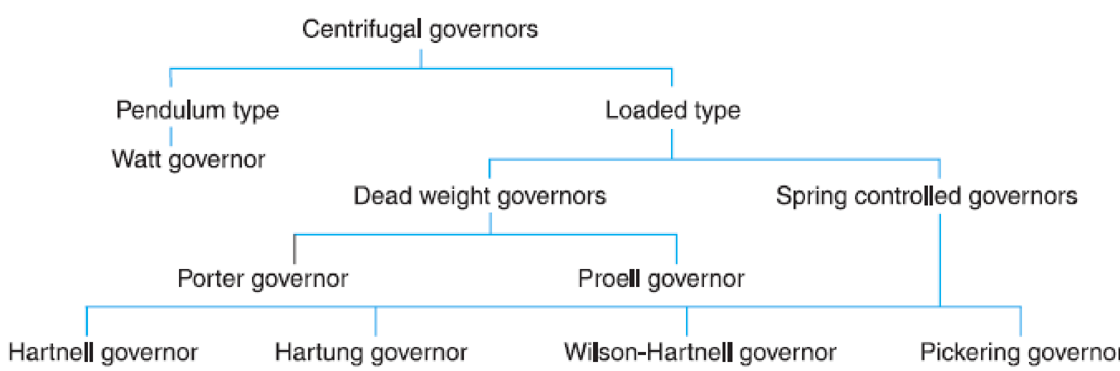
Flywheel

Flywheel is the machine member which avoids the fluctuation of energy in a power plant

Functions of Flywheel (2M Important)

- The function of the flywheel is to control speed variations caused by change in turning moment during a cycle.
- It stores energy and gives it out whenever required.
- It controls and regulates the speed only during one cycle
- It does not have any control over the quantity of charge supplied in the engine.

Types of Centrifugal Governor



Difference between Flywheel and Governor (Important 2M)

Flywheel	Governor
Flywheel reduces the fluctuation of speed during the thermodynamic cycles, but it does not maintain a constant speed.	Governor is a device to control the speed variation caused by the varying load.
The working of a flywheel does not depend upon the change in load or output required.	Governor operation depends upon the variation of load.
The operation of flywheels is continuous from cycle to cycle.	The operation of a governor is intermittent.
Speed control in a single cycle	Speed control over a period of time
The function of a flywheel is to store energy when mechanical energy is more than required for the operation and release the same when the available energy is less than required. Its inertia helps to run machines at a dead center.	The function of a governor is to regulate the fuel supply according to the load requirement and run the machine at a constant speed irrespective of the output required.
Do not have any control over the supply of charge or fuel.	Control the supply of fuel to the engine
It is relatively heavy and has large inertia force.	It's a light machine part
It is used in engines and fabricating machines such as punching machines, rolling mill, etc.	Governors are provided on engines and turbines.
It is desired where the fluctuation in input torque. e.g.: four stroke engine	Desired where the constant speed required e.g. Generator (there is even electronic governor for diesel generator)

Centrifugal Governors

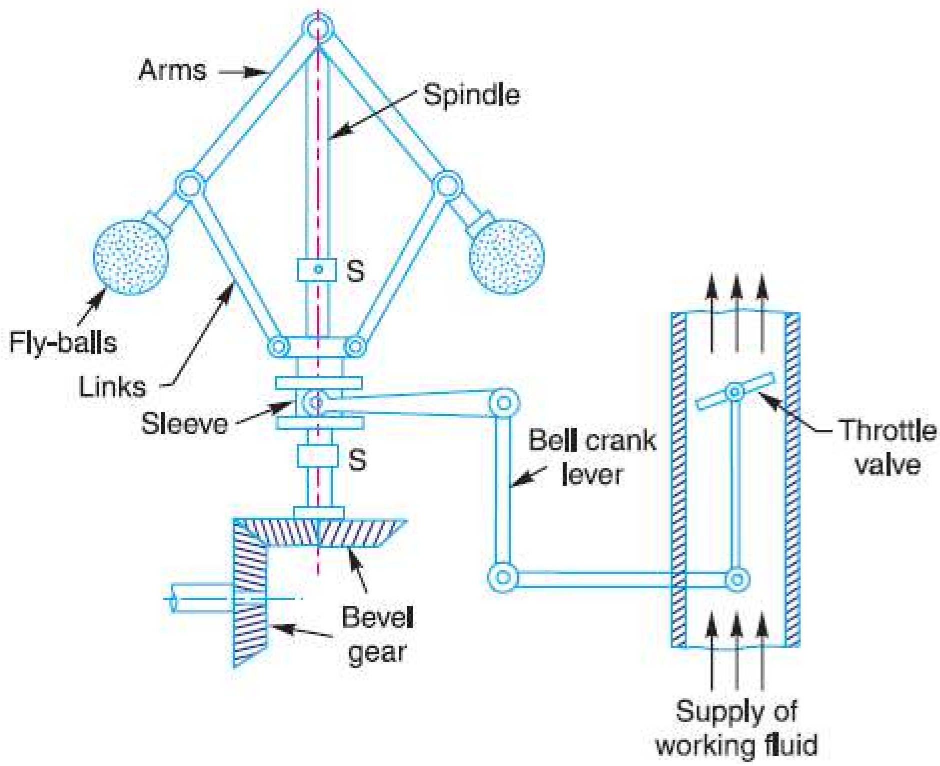


Fig. 18.1. Centrifugal governor.

The centrifugal governors are based on the balancing of centrifugal force on the rotating balls by an equal and opposite radial force, known as the ***controlling force****.It consists of two balls of equal mass, which are attached to the arms as shown in Fig. 18.1. These balls are known as ***governor balls* or *fly balls.*** The balls revolve with a spindle, which is driven by the engine through bevel gears. The upper ends of the arms are pivoted to the spindle, so that the balls may rise up or fall down as they revolve about the vertical axis. The arms are connected by the links to a sleeve, which is keyed to the spindle. This sleeve revolves with the spindle ; but can slide up and down. The balls and the sleeve rises when the spindle speed increases, and falls when the speed decreases. In order to limit the travel of the sleeve in upward and downward directions, two stops *S*, *S* are provided on the spindle. The sleeve is connected by a bell crank lever to a throttle valve. The supply of the working fluid decreases when the sleeve rises and increases when it falls. When the load on the engine increases, the engine and the governor speed decreases. This results in the decrease of centrifugal force on the balls. Hence the balls move inwards and the sleeve moves downwards. The downward movement of the sleeve operates a throttle valve at the other end of the bell crank lever to increase the supply of working fluid and thus the engine speed is increased. In this case, the extra power output is provided to balance the increased load. When the load on the engine decreases, the engine and the governor speed increases, which results in the increase of centrifugal force on the balls. Thus the balls move outwards and the sleeve rises upwards. This upward movement of the sleeve reduces the supply of the working fluid and hence the speed is decreased. In this case, the power output is reduced.

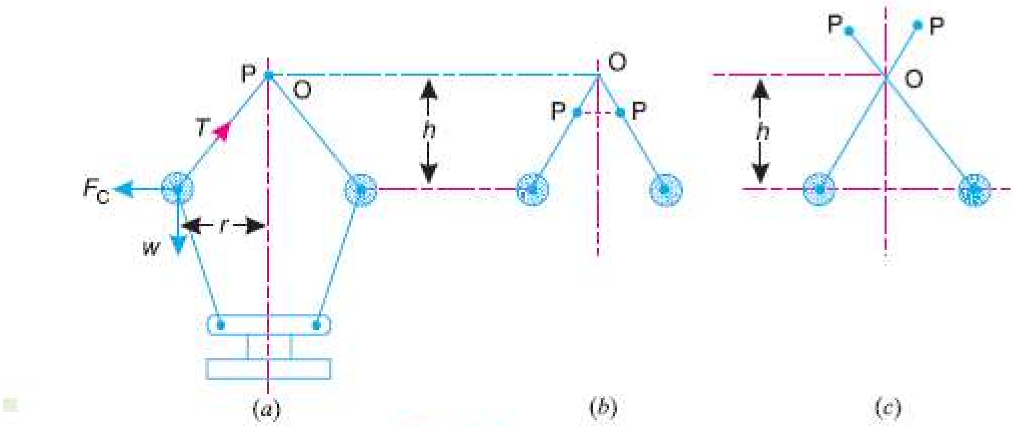
Watt Governor

The simplest form of a centrifugal governor is a Watt governor, as shown in Fig. 18.2. It is basically a conical pendulum with links attached to a sleeve of negligible mass. The arms of the governor may be connected to the spindle in the following three ways :

1. The pivot *P*, may be on the spindle axis as shown in Fig. 18.2 (*a*).

2. The pivot *P*, may be offset from the spindle axis and the arms when produced intersect at *O*, as shown in Fig. 18.2 (*b*).

3. The pivot *P*, may be offset, but the arms cross the axis at *O*, as shown in Fig. 18.2 (*c*).



m = Mass of the ball in kg,
w = Weight of the ball in newtons = $m.g$,
T = Tension in the arm in newtons,
ω = Angular velocity of the arm and ball about the spindle axis in rad/s,
r = Radius of the path of rotation of the ball *i.e.* horizontal distance from the centre of the ball to the spindle axis in metres,
F_C = Centrifugal force acting on the ball in newtons = $m.\omega^2.r$, and
h = Height of the governor in metres.

It is assumed that the weight of the arms, links and the sleeve are negligible as compared to the weight of the balls. Now, the ball is in equilibrium under the action of

1. the centrifugal force (F_C) acting on the ball, **2.** the tension (T) in the arm, and **3.** the weight (w) of the ball.

Taking moments about point O, we have

$$F_C \times h = w \times r = m.g.r$$

or $$m.\omega^2.r.h = m.g.r \quad \text{or} \quad h = g/\omega^2 \qquad \ldots (i)$$

When g is expressed in m/s² and ω in rad/s, then h is in metres. If N is the speed in r.p.m., then

$$\omega = 2\pi N/60$$

$$\therefore \quad h = \frac{9.81}{(2\pi N/60)^2} = \frac{895}{N^2} \text{ metres} \qquad \ldots (\because g = 9.81 \text{ m/s}^2) \ldots (ii)$$

Note : We see from the above expression that the height of a governor h, is inversely proportional to N^2. Therefore at high speeds, the value of h is small. At such speeds, the change in the value of h corresponding to a small change in speed is insufficient to enable a governor of this type to operate the mechanism to give the necessary change in the fuel supply. This governor may only work satisfactorily at relatively low speeds *i.e.* from 60 to 80 r.p.m.

Example 18.1. *Calculate the vertical height of a Watt governor when it rotates at 60 r.p.m. Also find the change in vertical height when its speed increases to 61 r.p.m.*

Solution. Given : N_1 = 60 r.p.m. ; N_2 = 61 r.p.m.

Initial height

We know that initial height,

Initial height

We know that initial height,

$$h_1 = \frac{895}{(N_1)^2} = \frac{895}{(60)^2} = 0.248 \text{ m}$$

Change in vertical height

We know that final height,

$$h_2 = \frac{895}{(N_2)^2} = \frac{895}{(61)^2} = 0.24 \text{ m}$$

$\therefore$ Change in vertical height

$$= h_1 - h_2 = 0.248 - 0.24 = 0.008 \text{ m} = 8 \text{ mm } \textbf{Ans.}$$

Porter Governor (10m Important)

The Porter governor is a modification of a Watt's governor, with central load attached to the sleeve as shown in Fig. 18.3 (*a*). The load moves up and down the central spindle. This additional downward force increases the speed of revolution required to enable the balls to rise to any predetermined level.

Consider the forces acting on one-half of the governor as shown in Fig. 18.3 (*b*)..

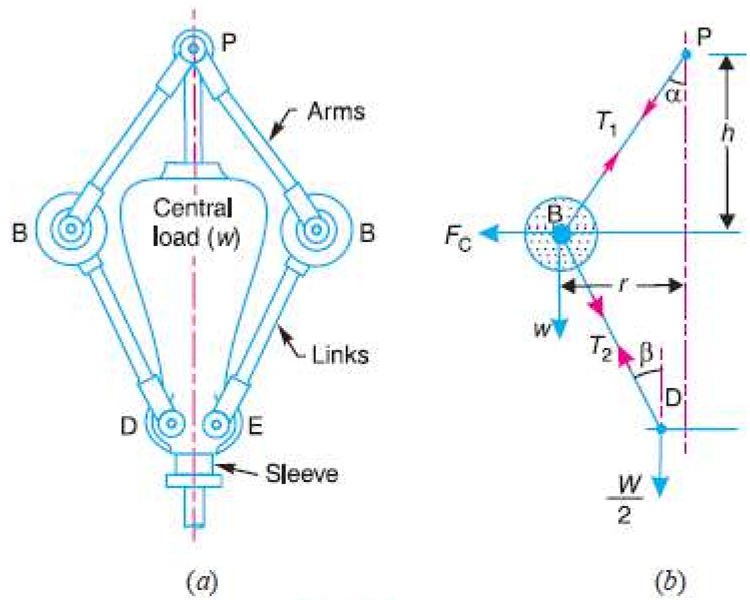


Fig. 18.3. Porter governor.

Let m = Mass of each ball in kg,

w = Weight of each ball in newtons = $m.g$,

M = Mass of the central load in kg,

W = Weight of the central load in newtons = $M.g$,

r = Radius of rotation in metres,

h = Height of governor in metres ,

N = Speed of the balls in r.p.m .,

ω = Angular speed of the balls in rad/s $= 2\pi N/60$ rad/s,

F_C = Centrifugal force acting on the ball in newtons $= m.\omega^2.r$,

T_1 = Force in the arm in newtons,

T_2 = Force in the link in newtons,

α = Angle of inclination of the arm (or upper link) to the vertical, and

β = Angle of inclination of the link (or lower link) to the vertical.

1. *Method of resolution of forces*

Considering the equilibrium of the forces acting at D, we have

$$T_2 \cos\beta = \frac{W}{2} = \frac{M.g}{2}$$

or
$$T_2 = \frac{M.g}{2\cos\beta} \quad \ldots (i)$$

Again, considering the equilibrium of the forces acting on B. The point B is in equilibrium under the action of the following forces, as shown in Fig. 18.3 (b).

(*i*) The weight of ball ($w = m.g$),

(*ii*) The centrifugal force (F_C),

(*iii*) The tension in the arm (T_1), and

(*iv*) The tension in the link (T_2).

Resolving the forces vertically,

$$T_1\cos\alpha = T_2\cos\beta + w = \frac{M.g}{2} + m.g \quad \ldots (ii)$$

$$\ldots \left(\because T_2\cos\beta = \frac{M.g}{2}\right)$$

Resolving the forces horizontally,

$$T_1\sin\alpha + T_2\sin\beta = F_C$$

$$T_1\sin\alpha + \frac{M.g}{2\cos\beta}\times\sin\beta = F_C \quad \ldots \left(\because T_2 = \frac{M.g}{2\cos\beta}\right)$$

$$T_1\sin\alpha + \frac{M.g}{2}\times\tan\beta = F_C$$

$$\therefore \quad T_1\sin\alpha = F_C - \frac{M.g}{2}\times\tan\beta \quad \ldots (iii)$$

Dividing equation (*iii*) by equation (*ii*),

$$\frac{T_1\sin\alpha}{T_1\cos\alpha} = \frac{F_C - \frac{M.g}{2}\times\tan\beta}{\frac{M.g}{2} + m.g}$$

or $$\left(\frac{M.g}{2} + m.g\right)\tan\alpha = F_C - \frac{M.g}{2} \times \tan\beta$$

$$\frac{M.g}{2} + m.g = \frac{F_C}{\tan\alpha} - \frac{M.g}{2} \times \frac{\tan\beta}{\tan\alpha}$$

Substituting $\frac{\tan\beta}{\tan\alpha} = q$, and $\tan\alpha = \frac{r}{h}$, we have

$$\frac{M.g}{2} + m.g = m.\omega^2.r \times \frac{h}{r} - \frac{M.g}{2} \times q \qquad \ldots (\because F_C = m.\omega^2.r)$$

or $$m.\omega^2.h = m.g + \frac{M.g}{2}(1+q)$$

$\therefore$ $$h = \left[m.g + \frac{M.g}{2}(1+q)\right]\frac{1}{m.\omega^2} = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{g}{\omega^2} \qquad \ldots (iv)$$

or $$\omega^2 = \left[m.g + \frac{Mg}{2}(1+q)\right]\frac{1}{m.h} = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{g}{h}$$

or $$\left(\frac{2\pi N}{60}\right)^2 = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{g}{h}$$

$$N^2 = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{g}{h}\left(\frac{60}{2\pi}\right)^2 = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{895}{h} \qquad \ldots (v)$$

... (Taking $g = 9.81$ m/s^2)

Notes : 1. When the length of arms are equal to the length of links and the points P and D lie on the same vertical line, then

$$\tan\alpha = \tan\beta \quad \text{or} \quad q = \tan\alpha / \tan\beta = 1$$

Therefore, the equation (*v*) becomes

$$N^2 = \frac{(m+M)}{m} \times \frac{895}{h} \qquad \ldots (vi)$$

2. When the loaded sleeve moves up and down the spindle, the frictional force acts on it in a direction opposite to that of the motion of sleeve.

If F = Frictional force acting on the sleeve in newtons, then the equations (*v*) and (*vi*) may be written as

$$N^2 = \frac{m.g + \left(\frac{M.g \pm F}{2}\right)(1+q)}{m.g} \times \frac{895}{h} \qquad \ldots (vii)$$

$$= \frac{m.g + (M.g \pm F)}{m.g} \times \frac{895}{h} \qquad \ldots (\text{When } q = 1) \ldots (viii)$$

2. Instantaneous centre method

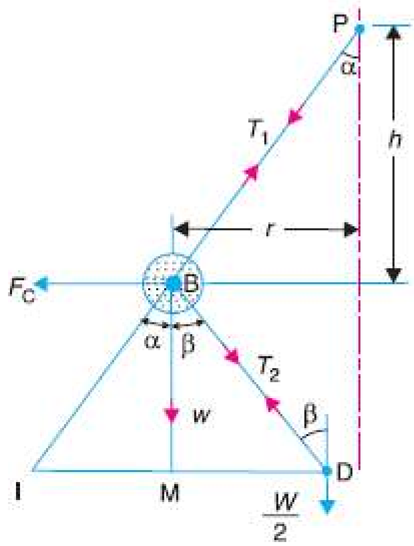


Fig. 18.4. Instantaneous centre method.

In this method, equilibrium of the forces acting on the link *BD* are considered. The instantaneous centre *I* lies at the point of intersection of *PB* produced and a line through *D* perpendicular to the spindle axis, as shown in Fig. 18.4. Taking moments about the point *I*,

$$F_C \times BM = w \times IM + \frac{W}{2} \times ID$$

$$= m.g \times IM + \frac{M.g}{2} \times ID$$

$$\therefore \quad F_C = m.g \times \frac{IM}{BM} + \frac{M.g}{2} \times \frac{ID}{BM}$$

$$= m.g \times \frac{IM}{BM} + \frac{M.g}{2}\left(\frac{IM + MD}{BM}\right)$$

$$= m.g \times \frac{IM}{BM} + \frac{M.g}{2}\left(\frac{IM}{BM} + \frac{MD}{BM}\right)$$

$$= m.g \tan\alpha + \frac{M.g}{2}(\tan\alpha + \tan\beta)$$

$$\ldots \left(\because \frac{IM}{BM} = \tan\alpha, \text{ and } \frac{MD}{BM} = \tan\beta\right)$$

Dividing throughout by tan α,

$$\frac{F_C}{\tan\alpha} = m.g + \frac{M.g}{2}\left(1 + \frac{\tan\beta}{\tan\alpha}\right) = m.g + \frac{M.g}{2}(1+q) \qquad \ldots\left(\because q = \frac{\tan\beta}{\tan\alpha}\right)$$

We know that $F_C = m.\omega^2.r,$ and $\tan\alpha = \frac{r}{h}$

$$\therefore\ m.\omega^2.r \times \frac{h}{r} = m.g + \frac{M.g}{2}(1+q)$$

or

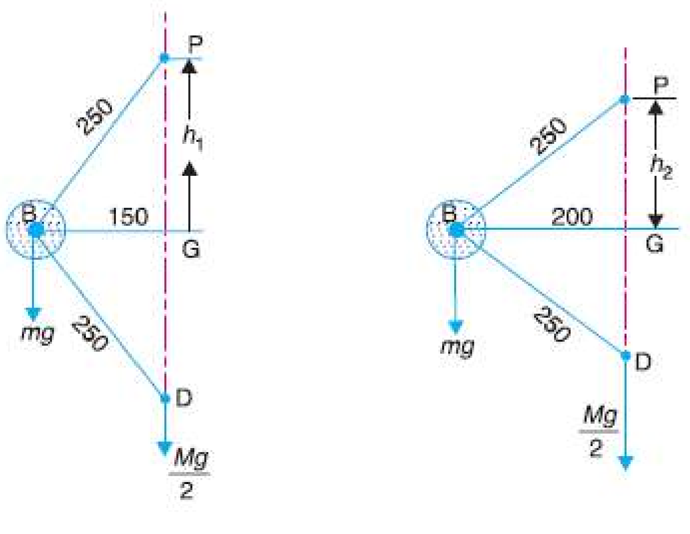
$$h = \frac{m.g + \frac{M.g}{2}(1+q)}{m} \times \frac{1}{\omega^2} = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{g}{\omega^2}$$

. . . (Same as before)

PROBLEMS

1. A Porter governor has equal arms each 250 mm long and pivoted on the axis of rotation. Each ball has a mass of 5 kg and the mass of the central load on the sleeve is 25 kg. The radius of rotation of the ball is 150 mm when the governor begins to lift and 200 mm when the governor is at maximum speed. Find the minimum and maximum speeds and range of speed of the governor.

Solution. Given : $BP = BD = 250$ mm = 0.25 m ; $m = 5$ kg ; $M = 15$ kg ; $r_1 = 150$ mm = 0.15m; $r_2 = 200$ mm = 0.2 m



(*a*) Minimum position. (*b*) Maximum position.

The minimum and maximum positions of the governor are shown in Fig. 18.5 (*a*) and (*b*) respectively.

Minimum speed when $r_1 = BG = 0.15$ m

Let N_1 = Minimum speed.

From Fig. 18.5 (*a*), we find that height of the governor,

$$h_1 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(0.25)^2 - (0.15)^2} = 0.2 \text{ m}$$

We know that

$$(N_1)^2 = \frac{m + M}{m} \times \frac{895}{h_1} = \frac{5 + 15}{5} \times \frac{895}{0.2} = 17\,900$$

$\therefore$ $N_1 = 133.8$ r.p.m. **Ans.**

Maximum speed when $r_2 = BG = 0.2$ m

Let N_2 = Maximum speed.

From Fig. 18.5 (*b*), we find that height of the governor,

$$h_2 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(0.25)^2 - (0.2)^2} = 0.15 \text{ m}$$

We know that

$$(N_2)^2 = \frac{m+M}{m} \times \frac{895}{h_2} = \frac{5+15}{5} \times \frac{895}{0.15} = 23\,867$$

$\therefore$ $N_2 = 154.5$ r.p.m. **Ans.**

Range of speed

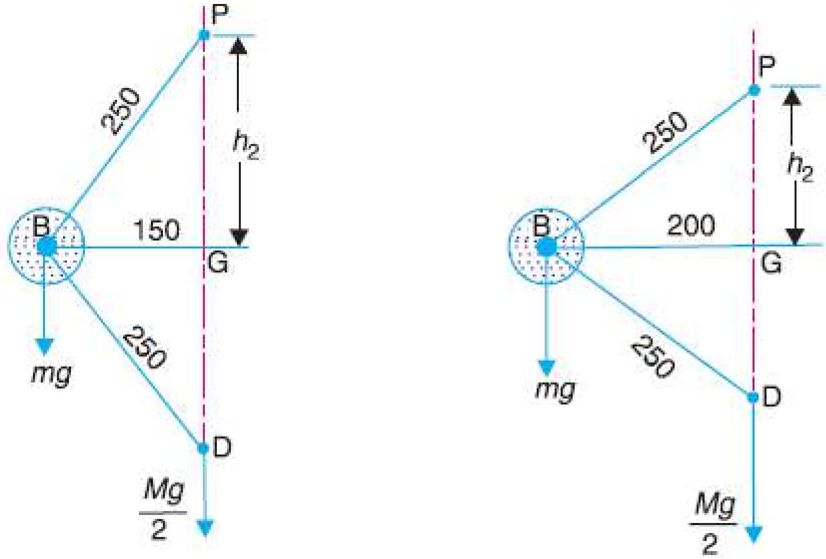
We know that range of speed

$$= N_2 - N_1 = 154.4 - 133.8 = 20.7 \text{ r.p.m. } \textbf{Ans.}$$

2. The arms of a Porter governor are each 250 mm long and pivoted on the governor axis. The mass of each ball is 5 kg and the mass of the central sleeve is 30 kg. The radius of rotation of the balls is 150 mm when the sleeve begins to rise and reaches a value of 200 mm for maximum speed. Determine the speed range of the governor. If the friction at the sleeve is equivalent of 20 N of load at the sleeve, determine how the speed range is modified. (IMPORTANT)

Solution. Given : $BP = BD = 250$ mm ; $m = 5$ kg ; $M = 30$ kg ; $r_1 = 150$ mm ; $r_2 = 200$ mm

First of all, let us find the minimum and maximum speed of the governor. The minimum and maximum position of the governor is shown in Fig. 18.6 (*a*) and (*b*) respectively. Let N_1 = Minimum speed when $r_1 = BG = 150$ mm, and N_2 = Maximum speed when $r_2 = BG$ = 200 mm.



(*a*) Minimum position. (*b*) Maximum position.

Speed range of the governor

From Fig. 18.6 (*a*), we find that height of the governor,

$$h_1 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(250)^2 - (150)^2} = 200 \text{ mm} = 0.2 \text{ m}$$

We know that

$$(N_1)^2 = \frac{m+M}{m} \times \frac{895}{h_1} = \frac{5+30}{5} \times \frac{895}{0.2} = 31\,325$$

$\therefore$ $N_1 = 177$ r.p.m.

From Fig. 18.6 (*b*), we find that height of the governor,

$$h_2 = PG = \sqrt{(PB)^2 - (BG)^2} = \sqrt{(250)^2 - (200)^2} = 150 \text{ mm} = 0.15 \text{ m}$$

We know that

$$(N_2)^2 = \frac{m+M}{m} \times \frac{895}{h_2} = \frac{5+30}{5} \times \frac{895}{0.15} = 41\,767$$

$\therefore$ $N_2 = 204.4$ r.p.m.

We know that speed range of the governor

$= N_2 - N_1 = 204.4 - 177 = 27.4$ r.p.m. **Ans.**

Speed range when friction at the sleeve is equivalent of 20 N of load (i.e. when F = 20 N)

We know that when the sleeve moves downwards, the friction force (F) acts upwards and the minimum speed is given by

$$(N_1)^2 = \frac{m.g + (M.g - F)}{m.g} \times \frac{895}{h_1}$$

$$= \frac{5 \times 9.81 + (30 \times 9.81 - 20)}{5 \times 9.81} \times \frac{895}{0.2} = 29\,500$$

$\therefore \qquad N_1 = 172$ r.p.m.

We also know that when the sleeve moves upwards, the frictional force (F) acts downwards and the maximum speed is given by

$$(N_2)^2 = \frac{m.g + (M.g + F)}{m.g} \times \frac{895}{h_2}$$

$$= \frac{5 \times 9.81 + (30 \times 9.81 + 20)}{5 \times 9.81} \times \frac{895}{0.15} = 44\,200$$

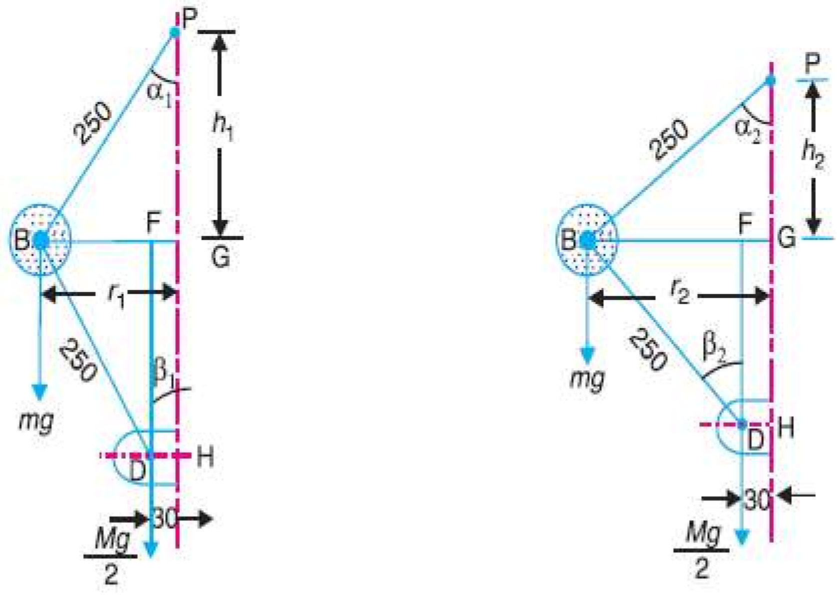
$\therefore \qquad N_2 = 210$ r.p.m.

We know that speed range of the governor

$= N_2 - N_1 = 210 - 172 = 38$ r.p.m. **Ans.**

3. *A Porter governor has all four arms 250 mm long. The upper arms are attached on the axis of rotation and the lower arms are attached to the sleeve at a distance of 30 mm from the axis. The mass of each ball is 5 kg and the sleeve has a mass of 50 kg. The extreme radii of rotation are 150 mm and 200 mm. Determine the range of speed of the governor.*

Solution. Given: *BP* = *BD* = 250 mm ; *DH* = 30 mm ; *m* = 5 kg ; *M* = 50 kg ; *r*1 = 150 mm ; *r*2 = 200 mm
First of all, let us find the minimum and maximum speed of the governor. The minimum and maximum position of the governor is shown in Fig. 18.8 (*a*) and (*b*) respectively.



(*a*) Minimum position. (*b*) Maximum position.

Let N_1 = Minimum speed when $r_1 = BG = 150$ mm ; and

N_2 = Maximum speed when $r_2 = BG = 200$ mm.

From Fig. 18.8 (*a*), we find that height of the governor,

$$h_1 = PG = \sqrt{(BP)^2 - (BG)^2} = \sqrt{(250)^2 - (150)^2} = 200 \text{ mm} = 0.2 \text{ m}$$

$$BF = BG - FG = 150 - 30 = 120 \text{ mm} \qquad \ldots (\because FG = DH)$$

and $$DF = \sqrt{(DB)^2 - (BF)^2} = \sqrt{(250)^2 - (120)^2} = 219 \text{ mm}$$

$\therefore$ $$\tan \alpha_1 = BG/PG = 150 / 200 = 0.75$$

and $$\tan \beta_1 = BF/DF = 120/219 = 0.548$$

$\therefore$ $$q_1 = \frac{\tan \beta_1}{\tan \alpha_1} = \frac{0.548}{0.75} = 0.731$$

We know that $$(N_1)^2 = \frac{m + \frac{M}{2}(1 + q_1)}{m} \times \frac{895}{h_1} = \frac{5 + \frac{50}{2}(1 + 0.731)}{5} \times \frac{895}{0.2} = 43\,206$$

$\therefore$ $$N_1 = 208 \text{ r.p.m.}$$

From Fig. 18.8(*b*), we find that height of the governor,

$$h_2 = PG = \sqrt{(BP)^2 - (BG)^2} = \sqrt{(250)^2 - (200)^2} = 150 \text{ mm} = 0.15 \text{ m}$$

$$BF = BG - FG = 200 - 30 = 170 \text{ mm}$$

and $$DF = \sqrt{(DB)^2 - (BF)^2} = \sqrt{(250)^2 - (170)^2} = 183 \text{ mm}$$

$\therefore$ $$\tan \alpha_2 = BG/PG = 200/150 = 1.333$$

and $$\tan \beta_2 = BF/DF = 170/183 = 0.93$$

$\therefore$ $$q_2 = \frac{\tan \beta_2}{\tan \alpha_2} = \frac{0.93}{1.333} = 0.7$$

We know that

$$(N_2)^2 = \frac{m + \frac{M}{2}(1 + q_2)}{m} \times \frac{895}{h_2} = \frac{5 + \frac{50}{2}(1 + 0.7)}{5} \times \frac{895}{0.15} = 56\,683$$

$\therefore$ $$N_2 = 238 \text{ r.p.m.}$$

We know that range of speed

$$= N_2 - N_1 = 238 - 208 = 30 \text{ r.p.m.}$$ **Ans.**

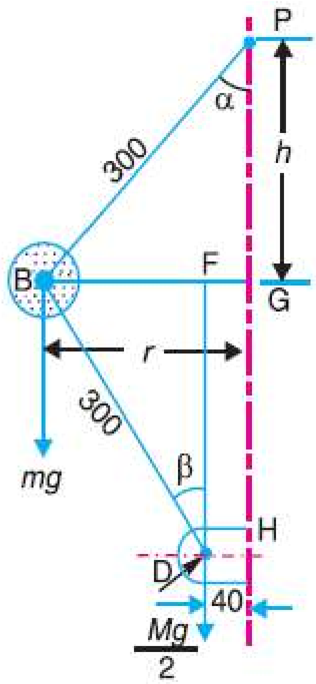
4. The arms of a Porter governor are 300 mm long. The upper arms are pivoted on the axis of rotation. The lower arms are attached to a sleeve at a distance of 40 mm from the axis of rotation. The mass of the load on the sleeve is 70 kg and the mass of each ball is 10 kg. Determine the equilibrium speed when the radius of rotation of the balls is 200 mm. If the friction is equivalent to a load of 20 N at the sleeve, what will be the range of speed for this position ?

Solution. Given : $BP = BD = 300$ mm ; $DH = 40$ mm ; $M = 70$ kg ; $m = 10$ kg ; $r = BG = 200$ mm

Equilibrium speed when the radius of rotation r = BG = 200 mm

Let N = Equilibrium speed.

. From the figure, we find that height of the governor,



h = PG = (BP)2 – (BG)2 = (300)2 – (200)2 = 224 mm = 0.224m

$BF = BG - FG$ = 200 – 40 = 160

and $$DF = \sqrt{(DB)^2 - (BF)^2} = \sqrt{(300)^2 - (160)^2} = 254 \text{ mm}$$

$\therefore \tan \alpha = BG/PG = 200 / 224 = 0.893$

and $\tan \beta = BF/DF = 160 / 254 = 0.63$

$$\therefore \quad q = \frac{\tan \beta}{\tan \alpha} = \frac{0.63}{0.893} = 0.705$$

We know that

$$N_2 = \frac{m + \frac{M}{2}(1 + q)}{m} \times \frac{895}{h}$$

$$= \frac{10 + \frac{70}{2}(1 + 0.705)}{10} \times \frac{895}{0.224} = 27\,840$$

N_2 = 167 r.p.m. **Ans.**

Range of speed when friction is equivalent to load of 20 N at the sleeve (i.e. when F = 20 N)

Let N_1 = Minimum equilibrium speed, and

N_2 = Maximum equilibrium speed.

We know that when the sleeve moves downwards, the frictional force (F) acts upwards and the minimum equilibrium speed is given by

$$(N_1)^2 = \frac{m.g + \left(\frac{M.g - F}{2}\right)(1+q)}{m.g} \times \frac{895}{h}$$

$$= \frac{10 \times 9.81 + \left(\frac{70 \times 9.81 - 20}{2}\right)(1+0.705)}{10 \times 9.81} \times \frac{895}{0.224} = 27\,144$$

$\therefore \quad N_1 = 164.8$ r.p.m.

We also know that when the sleeve moves upwards, the frictional force (F) acts downwards and the maximum equilibrium speed is given by

$$(N_2)^2 = \frac{m.g + \left(\frac{M.g + F}{2}\right)(1+q)}{m.g} \times \frac{895}{h}$$

$$= \frac{10 \times 9.81 + \left(\frac{70 \times 9.81 + 20}{2}\right)(1+0.705)}{10 \times 9.81} \times \frac{895}{0.224} = 28\,533$$

$\therefore \quad N_2 = 169$ r.p.m.

Proell Governor

The Proell governor has the balls fixed at B and C to the extension of the links DF and EG, as shown in Fig. 18.12 (*a*). The arms FP and GQ are pivoted at P and Q respectively.

Consider the equilibrium of the forces on one-half of the governor as shown in Fig. 18.12 (*b*). The instantaneous centre (I) lies on the intersection of the line PF produced and the line from D drawn perpendicualr to the spindle axis. The prependicular BM is drawn on ID.

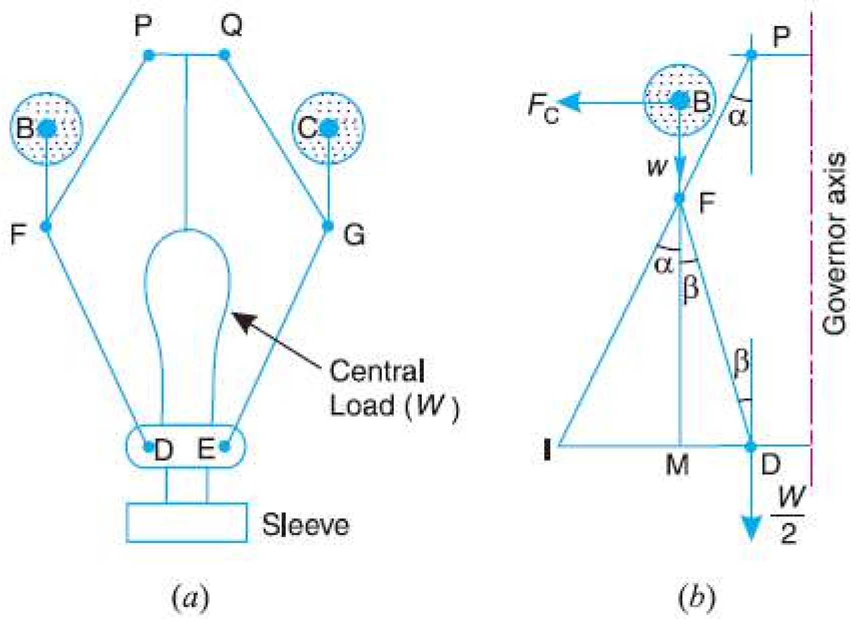


Fig. 18.12. Proell governor.

Taking moments about *I*, using the same notations as discussed in Art. 18.6 (Porter governor),

$$F_C \times BM = w \times IM + \frac{W}{2} \times ID = m.g \times IM + \frac{M.g}{2} \times ID \quad \ldots (i)$$

$$\therefore \quad F_C = m.g \times \frac{IM}{BM} + \frac{M.g}{2}\left(\frac{IM + MD}{BM}\right) \quad \ldots (\because \; ID = IM + MD)$$

Multiplying and dividing by *FM*, we have

$$F_C = \frac{FM}{BM}\left[m.g \times \frac{IM}{FM} + \frac{M.g}{2}\left(\frac{IM}{FM} + \frac{MD}{FM}\right)\right]$$

$$= \frac{FM}{BM}\left[m.g \times \tan\alpha + \frac{M.g}{2}(\tan\alpha + \tan\beta)\right]$$

$$= \frac{FM}{BM} \times \tan\alpha \left[m.g + \frac{M.g}{2}\left(1 + \frac{\tan\beta}{\tan\alpha}\right)\right]$$

We know that $F_C = m.\omega^2 r$; $\tan\alpha = \frac{r}{h}$ and $q = \frac{\tan\beta}{\tan\alpha}$

$$\therefore \qquad m.\omega^2.r = \frac{FM}{BM} \times \frac{r}{h}\left[m.g + \frac{M.g}{2}(1+q)\right]$$

and

$$\omega^2 = \frac{FM}{BM}\left[\frac{m + \frac{M}{2}(1+q)}{m}\right]\frac{g}{h}$$

Substituting $\omega = 2\pi N/60$, and $g = 9.81$ m/s², we get

$$N^2 = \frac{FM}{BM}\left[\frac{m + \frac{M}{2}(1+q)}{m}\right]\frac{895}{h}$$

Notes : 1. The equation *(i)* may be applied to any given configuration of the governor.

2. Comparing equation *(iii)* with the equation *(v)* of the Porter governor (Art. 18.6), we see that the equilibrium speed reduces for the given values of m, M and h. Hence in order to have the same equilibrium speed for the given values of m, M and h, balls of smaller masses are used in the Proell governor than in the Porter governor.

3. When $\alpha = \beta$, then $q = 1$. Therefore equation *(iii)* may be written as

$$N^2 = \frac{FM}{BM}\left(\frac{m+M}{m}\right)\frac{895}{h} \qquad (h \text{ being in metres}) \quad ...(iv)$$

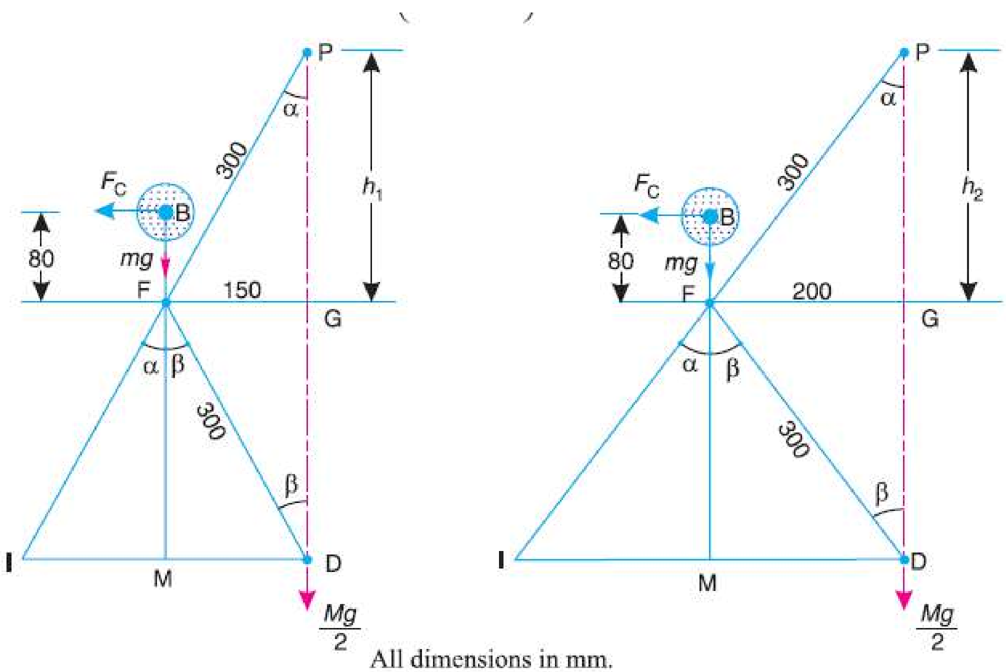
1 A Proell governor has equal arms of length 300 mm. The upper and lower ends of the arms are pivoted on the axis of the governor. The extension arms of the lower links are each 80 mm long and parallel to the axis when the radii of rotation of the balls are 150 mm and 200 mm. The mass of each ball is 10 kg and the mass of the central load is 100 kg. Determine the range of speed of the governor.

Solution. Given : $PF = DF = 300$ mm ; $BF = 80$ mm ; $m = 10$ kg ; $M = 100$ kg ; $r_1 = 150$ mm; $r_2 = 200$ mm

First of all, let us find the minimum and maximum speed of the governor. The minimum and maximum position of the governor is shown in Fig. 18.13.

Let N_1 = Minimum speed when radius of rotation, $r_1 = FG = 150$ mm ; and

N_2 = Maximum speed when radius of rotation , $r_2 = FG = 200$ mm.



(*a*) Minimum position. (*a*) Maximum position.

Fig. 18.13

From Fig. 18.13 (*a*), we find that height of the governor,

$$h_1 = PG = \sqrt{(PF)^2 - (FG)^2} = \sqrt{(300)^2 - (150)^2} = 260 \text{ mm} = 0.26 \text{ m}$$

and $FM = GD = PG = 260 \text{ mm} = 0.26 \text{ m}$

$\therefore$ $BM = BF + FM = 80 + 260 = 340 \text{ mm} = 0.34 \text{ m}$

We know that $(N_1)^2 = \dfrac{FM}{BM}\left(\dfrac{m+M}{m}\right)\dfrac{895}{h_1}$ . . . ($\because \alpha = \beta$ or $q = 1$)

$$= \frac{0.26}{0.34}\left(\frac{10+100}{10}\right)\frac{895}{0.26} = 28\,956 \quad \text{or} \quad N_1 = 170 \text{ r.p.m.}$$

Now from Fig. 18.13 (*b*), we find that height of the governor,

$$h_2 = PG = \sqrt{(PF)^2 - (FG)^2} = \sqrt{(300)^2 - (200)^2} = 224 \text{ mm} = 0.224 \text{ m}$$

and $FM = GD = PG = 224 \text{ mm} = 0.224 \text{ m}$

$\therefore$ $BM = BF + FM = 80 + 224 = 304 \text{ mm} = 0.304 \text{ m}$

We know that $(N_2)^2 = \dfrac{FM}{BM}\left(\dfrac{m+M}{m}\right)\dfrac{895}{h_2}$ . . . ($\because \alpha = \beta$ or $q = 1$)

$$= \frac{0.224}{0.304}\left(\frac{10+100}{10}\right)\frac{895}{0.224} = 32\,385 \quad \text{or} \quad N_2 = 180 \text{ r.p.m.}$$

We know that range of speed

$= N_2 - N_1 = 180 - 170 = 10$ r.p.m. **Ans.**

Hartnell Governor

A Hartnell governor is a spring loaded governor as shown in Fig. 18.18. It consists of two bell crank levers pivoted at the points O,O to the frame. The frame is attached to the governor spindle and therefore rotates with it. Each lever carries a ball at the end of the vertical arm OB and a roller at the end of the horizontal arm OR. A helical spring in compression provides equal downward forces on the two rollers through a collar on the sleeve. The spring force may be adjusted by screwing a nut up or down on the sleeve.

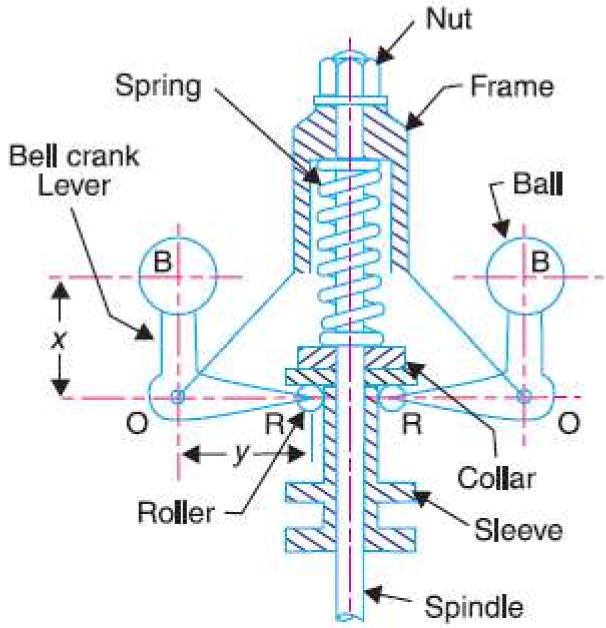


Fig. 18.18. Hartnell governor.

Let m = Mass of each ball in kg,

M = Mass of sleeve in kg,

r_1 = Minimum radius of rotation in metres,

r_2 = Maximum radius of rotation in metres,

ω_1 = Angular speed of the governor at minimum radius in rad/s,

ω_2 = Angular speed of the governor at maximum radius in rad/s,

S_1 = Spring force exerted on the sleeve at ω_1 in newtons,

S_2 = Spring force exerted on the sleeve at ω_2 in newtons,

F_{C1} = Centrifugal force at ω_1 in newtons = $m\ (\omega_1)^2\ r_1$,

F_{C2} = Centrifugal force at ω_2 in newtons = $m\ (\omega_2)^2\ r_2$,

s = Stiffness of the spring or the force required to compress the spring by one mm,

x = Length of the vertical or ball arm of the lever in metres,

y = Length of the horizontal or sleeve arm of the lever in metres, and

r = Distance of fulcrum O from the governor axis or the radius of rotation when the governor is in mid-position, in metres.

Consider the forces acting at one bell crank lever. The minimum and maximum position is shown in Fig. 18.19. Let h be the compression of the spring when the radius of rotation changes from r_1 to r_2.

For the minimum position *i.e.* when the radius of rotation changes from r to r_1, as shown in Fig. 18.19 (*a*), the compression of the spring or the lift of sleeve h_1 is given by

$$\frac{h_1}{y} = \frac{a_1}{x} = \frac{r - r_1}{x} \quad \ldots (i)$$

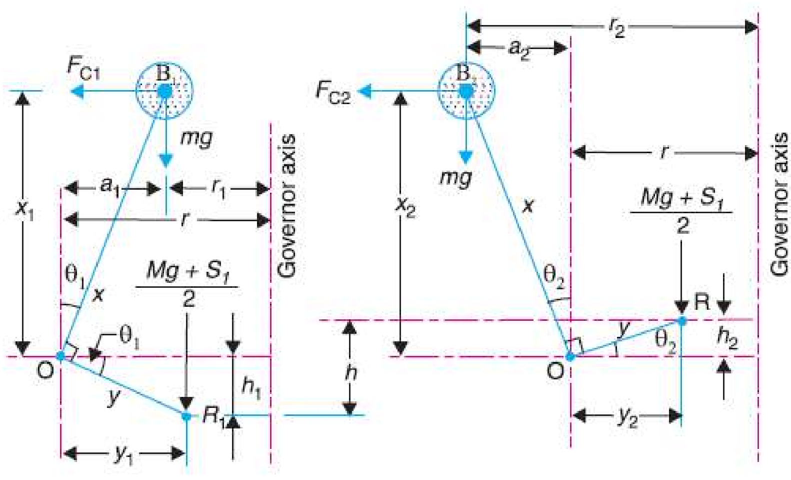
Similarly, for the maximum position *i.e.* when the radius of rotation changes from r to r_2, as shown in Fig. 18.19 (*b*), the compression of the spring or lift of sleeve h_2 is given by

$$\frac{h_2}{y} = \frac{a_2}{x} = \frac{r_2 - r}{x} \quad \ldots (ii)$$

Adding equations (*i*) and (*ii*),

$$\frac{h_1 + h_2}{y} = \frac{r_2 - r_1}{x} \quad \text{or} \quad \frac{h}{y} = \frac{r_2 - r_1}{x} \quad \ldots (\because\ h = h_1 + h_2)$$

$$\therefore \quad h = (r_2 - r_1)\,\frac{y}{x} \quad \ldots (iii)$$



(*a*) Minimum position. (*b*) Maximum position.

Fig. 18.19

Now for minimum position, taking moments about point O, we get

$$\frac{M.g + S_1}{2} \times y_1 = F_{C1} \times x_1 - m.g \times a_1$$

or $$M.g + S_1 = \frac{2}{y_1}(F_{C1} \times x_1 - m.g \times a_1) \qquad \ldots (iv)$$

Again for maximum position, taking moments about point O, we get

$$\frac{M.g + S_2}{2} \times y_2 = F_{C2} \times x_2 + m.g \times a_2$$

or $$M.g + S_2 = \frac{2}{y_2}(F_{C2} \times x_2 + m.g \times a_2) \qquad \ldots (v)$$

Subtracting equation *(iv)* from equation *(v)*,

$$S_2 - S_1 = \frac{2}{y_2}(F_{C2} \times x_2 + m.g \times a_2) - \frac{2}{y_1}(F_{C1} \times x_1 - m.g \times a_1)$$

We know that

$$S_2 - S_1 = h.s, \qquad \text{and} \qquad h = (r_2 - r_1)\frac{y}{x}$$

$$\therefore \qquad s = \frac{S_2 - S_1}{h} = \left(\frac{S_2 - S_1}{r_2 - r_1}\right)\frac{x}{y}$$

Neglecting the obliquity effect of the arms (*i.e.* $x_1 = x_2 = x$, and $y_1 = y_2 = y$) and the moment due to weight of the balls (*i.e.* $m.g$), we have for minimum position,

$$\frac{M.g + S_1}{2} \times y = F_{C1} \times x \qquad \text{or} \qquad M.g + S_1 = 2F_{C1} \times \frac{x}{y} \qquad \ldots (vi)$$

Similarly for maximum position,

$$\frac{M.g + S_2}{2} \times y = F_{C2} \times x \qquad \text{or} \qquad M.g + S_2 = 2F_{C2} \times \frac{x}{y} \qquad \ldots (vii)$$

Subtracting equation *(vi)* from equation *(vii)*,

$$S_2 - S_1 = 2(F_{C2} - F_{C1})\frac{x}{y} \qquad \ldots (viii)$$

We know that

$$S_2 - S_1 = h.s, \qquad \text{and} \qquad h = (r_2 - r_1)\frac{y}{x}$$

$$\therefore \qquad s = \frac{S_2 - S_1}{h} = 2\left(\frac{F_{C2} - F_{C1}}{r_2 - r_1}\right)\left(\frac{x}{y}\right)^2 \qquad \ldots (ix)$$

Notes : 1. Unless otherwise stated, the obliquity effect of the arms and the moment due to the weight of the balls is neglected, in actual practice.

2. When friction is taken into account, the weight of the sleeve ($M.g$) may be replaced by ($M.g. \pm F$).

3. The centrifugal force (F_C) for any intermediate position (*i.e.* between the minimum and maximum position) at a radius of rotation (r) may be obtained as discussed below :

Since the stiffness for a given spring is constant for all positions, therefore for minimum and intermediate position,

$$s = 2\left(\frac{F_C - F_{C1}}{r - r_1}\right)\left(\frac{x}{y}\right)^2 \qquad \ldots (x)$$

and for intermediate and maximum position,

$$s = 2\left(\frac{F_{C2} - F_C}{r_2 - r}\right)\left(\frac{x}{y}\right)^2 \quad \ldots (xi)$$

∴ From equations (*ix*), (*x*) and (*xi*),

$$\frac{F_{C2} - F_{C1}}{r_2 - r_1} = \frac{F_C - F_{C1}}{r - r_1} = \frac{F_{C2} - F_C}{r_2 - r}$$

or

$$F_C = F_{C1} + (F_{C2} - F_{C1})\left(\frac{r - r_1}{r_2 - r_1}\right) = F_{C2} - (F_{C2} - F_{C1})\left(\frac{r_2 - r}{r_2 - r_1}\right)$$

1. A Hartnell governor having a central sleeve spring and two right-angled bell crank levers moves between 290 r.p.m. and 310 r.p.m. for a sleeve lift of 15mm. The sleeve arms and the ball arms are 80 mm and 120 mm respectively. The levers are pivoted at 120 mm from the governor axis and mass of each ball is 2.5 kg. The ball arms are parallel to the governor axis at the lowest equilibrium speed. Determine : **1.** loads on the spring at the lowest and the highest equilibrium speeds, and **2.** stiffness of the spring.(IMPORTANT)(10M)

Solution. Given : $N_1 = 290$ r.p.m. or $\omega_1 = 2\pi \times 290/60 = 30.4$ rad/s ; $N_2 = 310$ r.p.m. or $\omega_2 = 2\pi \times 310/60 = 32.5$ rad/s ; $h = 15$ mm $= 0.015$ m ; $y = 80$ mm $= 0.08$ m ; $x = 120$ mm $= 0.12$ m ; $r = 120$ mm $= 0.12$ m ; $m = 2.5$ kg

1. Loads on the spring at the lowest and highest equilibrium speeds

Let S = Spring load at lowest equilibrium speed, and

S_2 = Spring load at highest equilibrium speed.

Since the ball arms are parallel to governor axis at the lowest equilibrium speed (*i.e.* at $N_1 = 290$ r.p.m.), as shown in Fig. 18.20 (*a*), therefore

$$r = r_1 = 120 \text{ mm} = 0.12 \text{ m}$$

We know that centrifugal force at the minimum speed,

$$F_{C1} = m\,(\omega_1)^2\, r_1 = 2.5\,(30.4)^2\; 0.12 = 277 \text{ N}$$

Now let us find the radius of rotation at the highest equilibrium speed, *i.e.* at $N_2 = 310$ r.p.m. The position of ball arm and sleeve arm at the highest equilibrium speed is shown in Fig. 18.20 (*b*).

Let r_2 = Radius of rotation at $N_2 = 310$ r.p.m.

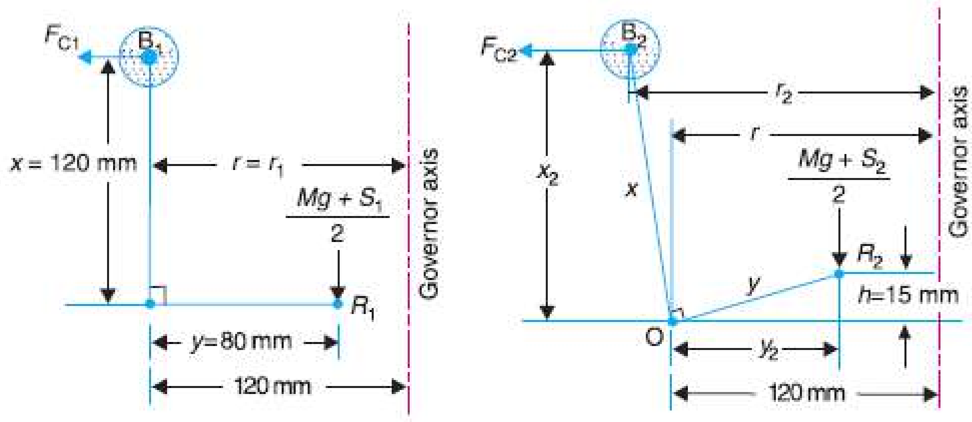
We know that $h = (r_2 - r_1)\,\dfrac{y}{x}$

or

$$r_2 = r_1 + h\left(\frac{x}{y}\right) = 0.12 + 0.015\left(\frac{0.12}{0.08}\right) = 0.1425 \text{ m}$$

∴ Centrifugal force at the maximum speed,

$$F_{C2} = m\,(\omega_2)^2\, r_2 = 2.5 \times (32.5)^2 \times 0.1425 = 376 \text{ N}$$



(*a*) Lowest position.

(*b*) Highest position.

Fig. 18.20

Neglecting the obliquity effect of arms and the moment due to the weight of the balls, we have for lowest position,

$$M.g + S_1 = 2F_{C1} \times \frac{x}{y} = 2 \times 277 \times \frac{0.12}{0.08} = 831 \text{ N}$$

$\therefore$ $S_2 = 831$ N **Ans.** $(\because M = 0)$

and for highest position,

$$M.g + S_2 = 2F_{C2} \times \frac{x}{y} = 2 \times 376 \times \frac{0.12}{0.08} = 1128 \text{ N}$$

$\therefore$ $S_1 = 1128$ N **Ans.** $(\because M = 0)$

2. Stiffness of the spring

We know that stiffness of the spring,

$$s = \frac{S_2 - S_1}{h} = \frac{1128 - 831}{15} = 19.8 \text{ N/mm} \quad \textbf{Ans.}$$

2. In a spring loaded Hartnell type governor, the extreme radii of rotation of the balls are 80 mm and 120 mm. The ball arm and the sleeve arm of the bell crank lever are equal in length. The mass of each ball is 2 kg. If the speeds at the two extreme positions are 400 and 420 r.p.m., find : **1.** the initial compression of the central spring, and **2.** the spring constant.

Solution. Given : $r_1 = 80$ mm $= 0.08$ m ; $r_2 = 120$ mm $= 0.12$ m ; $x = y$; $m = 2$ kg ; $N_1 = 400$ r.p.m. or $\omega = 2\pi \times 400/60 = 41.9$ rad/s ; $N_2 = 420$ r.p.m. or $\omega_2 = 2\pi \times 420/60 = 44$ rad/s

Initial compression of the central spring

We know that the centrifugal force at the minimum speed,

$$F_{C1} = m(\omega_1)^2 r_1 = 2(41.9)^2 \; 0.08 = 281 \text{ N}$$

and centrifugal force at the maximum speed,

$$F_{C2} = m(\omega_2)^2 r_2 = 2(44)^2 \; 0.12 = 465 \text{ N}$$

Let S_1 = Spring force at the minimum speed, and

S_2 = Spring force at the maximum speed.

We know that for minimum position,

$$M.g + S_1 = 2F_{C1} \times \frac{x}{y}$$

$\therefore$ $S_1 = 2F_{C1} = 2 \times 281 = 562$ N $\quad \ldots(\because M = 0 \text{ and } x = y)$

Similarly for maximum position,

$$M.g + S_2 = 2\,F_{C2} \times \frac{x}{y}$$

$\therefore$ $$S_2 = 2\,F_{C2} = 2 \times 465 = 930 \text{ N}$$

We know that lift of the sleeve,

$$h = (r_2 - r_1)\frac{y}{x} = r_2 - r_1 = 120 - 80 = 40 \text{ mm} \qquad \ldots (\because x = y)$$

$\therefore$ Stiffness of the spring,

$$s = \frac{S_2 - S_1}{h} = \frac{930 - 562}{40} = 9.2 \text{ N/mm}$$

We know that initial compression of the central spring

$$= \frac{S_1}{s} = \frac{562}{9.2} = 61 \text{ mm} \textbf{ Ans.}$$

2. ***Spring constant***

We have calculated above that the spring constant or stiffness of the spring,

$$s = 9.2 \text{ N/mm} \textbf{ Ans.}$$

Hartung Governor

A spring controlled governor of the Hartung type is shown in Fig. 18.26 (*a*). In this type of governor, the vertical arms of the bell crank levers are fitted with spring balls which compress against the frame of the governor when the rollers at the horizontal arm press against the sleeve.

Let S = Spring force,

F_C = Centrifugal force,

M = Mass on the sleeve, and

x and y = Lengths of the vertical and horizontal arm of the bell crank lever respectively.

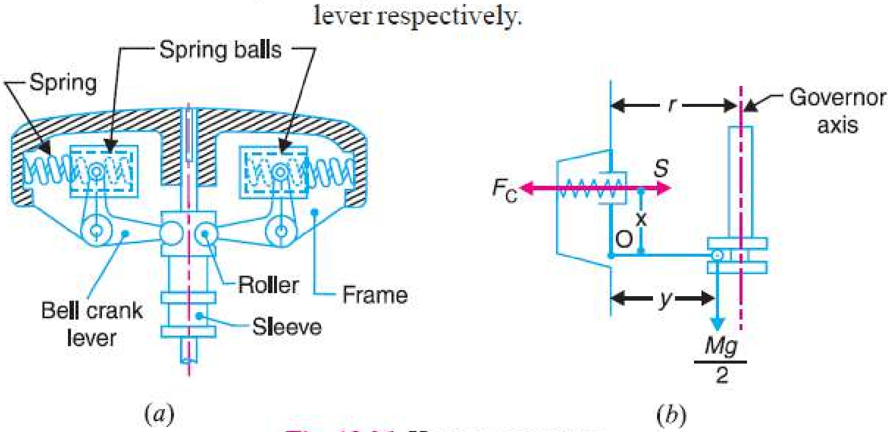


Fig. 18.26. Hartung governor.

Fig. 18.26 (*a*) and (*b*) show the governor in mid-position. Neglecting the effect of obliquity of the arms, taking moments about the fulcrum *O*,

$$F_C \times x = S \times x + \frac{M.g}{2} \times y$$

Sensitiveness of Governors (imp 2 M)

sensitiveness is defined as the ratio of the difference between the maximum and minimum equilibrium speeds to the mean equilibrium speed.

Let N_1 = Minimum equilibrium speed,

N_2 = Maximum equilibrium speed, and

N = Mean equilibrium speed $= \dfrac{N_1 + N_2}{2}$.

∴ Sensitiveness of the governor

$$= \frac{N_2 - N_1}{N} = \frac{2(N_2 - N_1)}{N_1 + N_2}$$

$$= \frac{2(\omega_2 - \omega_1)}{\omega_1 + \omega_2} \quad \text{. . . (In terms of angular speeds)}$$

Isochronous Governors (imp 2 M)

A governor is said to be ***isochronous*** when the equilibrium speed is constant (*i.e.* range of speed is zero) for all radii of rotation of the balls within the working range, neglecting friction. The isochronism is the stage of infinite sensitivity.

Let us consider the case of a Porter governor running at speeds N_1 and N_2 r.p.m. We have discussed in Art. 18.6 that

$$(N_1)^2 = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{895}{h_1} \quad \text{. . . } (i)$$

and

$$(N_2)^2 = \frac{m + \frac{M}{2}(1+q)}{m} \times \frac{895}{h_2} \quad \text{. . . } (ii)$$

For isochronism, range of speed should be zero *i.e.* $N_2 - N_1 = 0$ or $N_2 = N_1$. Therefore from equations (*i*) and (*ii*), $h_1 = h_2$, which is impossible in case of a Porter governor. Hence a ***Porter governor cannot be isochronous.***

Hunting (imp 2 M)

A governor is said to be **hunt** if the speed of the engine fluctuates continuously above and below the mean speed. This is caused by a too sensitive governor which changes the fuel supply by a large amount when a small change in the speed of rotation takes place.

Effort and Power of a Governor (imp 2 M)

The **effort of a governor** is the mean force exerted at the sleeve for a given percentage change of speed.

BALANCING
OF
ROTATING MASSES

Notes Compiled by:

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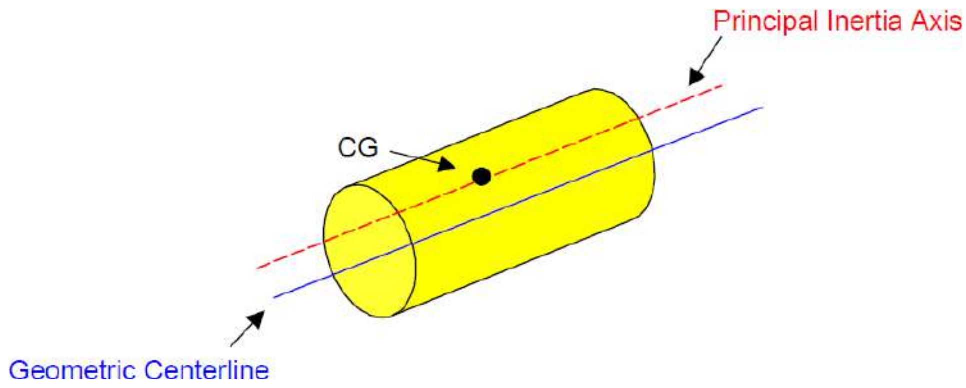
INTRODUCTION:

When man invented the wheel, he very quickly learnt that if it wasn"t completely round and if it didn"t rotate evenly about it"s central axis, then he had a problem!
What the problem he had?
The wheel would vibrate causing damage to itself and it"s support mechanism and in severe cases, is unusable.
A method had to be found to minimize the problem. The mass had to be evenly distributed about the rotating centerline so that the resultant vibration was at a minimum.

UNBALANCE:

The condition which exists in a rotor when vibratory force or motion is imparted to its bearings as a result of centrifugal forces is called unbalance or the uneven distribution of mass about a rotor"s rotating centerline.



Rotating centerline:

The rotating centerline being defined as the axis about which the rotor would rotate if not constrained by its bearings. (Also called the Principle Inertia Axis or PIA).

Geometric centerline:

The geometric centerline being the physical centerline of the rotor.

When the two centerlines are coincident, then the rotor will be in a state of balance. When they are apart, the rotor will be unbalanced.

Different types of unbalance can be defined by the relationship between the two centerlines. These include:
Static Unbalance – where the PIA is displaced parallel to the geometric centerline.

(Shown above)
Couple Unbalance – where the PIA intersects the geometric centerline at the center of gravity. (CG)
Dynamic Unbalance – where the PIA and the geometric centerline do not coincide or touch.

The most common of these is dynamic unbalance.

Causes of Unbalance:

In the design of rotating parts of a machine every care is taken to eliminate any out of balance or couple, but there will be always some residual unbalance left in the finished part because of

1. slight variation in the density of the material or
2. inaccuracies in the casting or
3. inaccuracies in machining of the parts.

Why balancing is so important?

1. A level of unbalance that is acceptable at a low speed is completely unacceptable at a higher speed.
2. As machines get bigger and go faster, the effect of the unbalance is much more severe.
3. The force caused by unbalance increases by the square of the speed.
4. If the speed is doubled, the force quadruples; if the speed is tripled the force increases

by a factor of nine!

Identifying and correcting the mass distribution and thus minimizing the force and resultant vibration is very very important

BALANCING:

Balancing is the technique of correcting or eliminating unwanted inertia forces or moments in rotating or reciprocating masses and is achieved by changing the location of the mass centers.
The objectives of balancing an engine are to ensure:

1. That the centre of gravity of the system remains stationery during a complete revolution of the crank shaft and
2. That the couples involved in acceleration of the different moving parts balance each other.

Types of balancing:

a) Static Balancing:

i) Static balancing is a balance of forces due to action of gravity.
ii) A body is said to be in static balance when its centre of gravity is in the axis of rotation.

b) Dynamic balancing:

i) Dynamic balance is a balance due to the action of inertia forces.
ii) A body is said to be in dynamic balance when the resultant moments or couples, which involved in the acceleration of different moving parts is equal to zero.
iii) The conditions of dynamic balance are met, the conditions of static balance are also met.

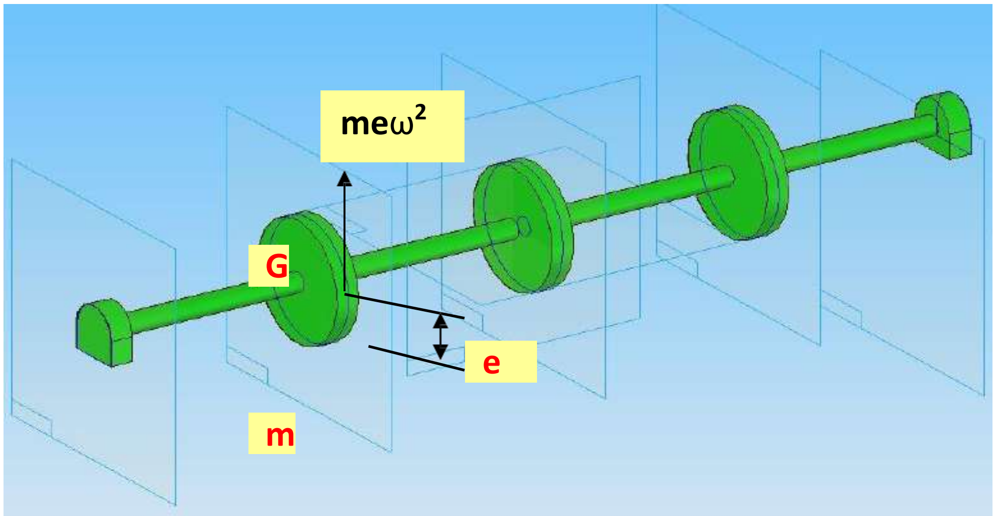
In rotor or reciprocating machines many a times unbalance of forces is produced due to inertia forces associated with the moving masses. If these parts are not properly balanced, the dynamic forces are set up and forces not only increase loads on bearings and stresses in the various components, but also unpleasant and dangerous vibrations.

Balancing is a process of designing or modifying machinery so that the unbalance is reduced to an acceptable level and if possible eliminated entirely.

BALANCING OF ROTATING MASSES

When a mass moves along a circular path, it experiences a centripetal acceleration and a force is required to produce it. An equal and opposite force called centrifugal force acts radially outwards and is a disturbing force on the axis of rotation. The magnitude of this remains constant but the direction changes with the rotation of the mass.

In a revolving rotor, the centrifugal force remains balanced as long as the centre of the mass of rotor lies on the axis of rotation of the shaft. When this does not happen, there is an eccentricity and an unbalance force is produced. This type of unbalance is common in steam turbine rotors, engine crankshafts, rotors of compressors, centrifugal pumps etc.



The unbalance forces exerted on machine members are time varying, impart vibratory motion and noise, there are human discomfort, performance of the machine deteriorate and detrimental effect on the structural integrity of the machine foundation.

Balancing involves redistributing the mass which may be carried out by addition or removal of mass from various machine members Balancing of rotating masses can be of

1. Balancing of a single rotating mass by a single mass rotating in the same plane.
2. Balancing of a single rotating mass by two masses rotating in different planes.
3. Balancing of several masses rotating in the same plane
4. Balancing of several masses rotating in different planes

STATIC BALANCING:

A system of rotating masses is said to be in static balance if the combined mass centre of the system lies on the axis of rotation

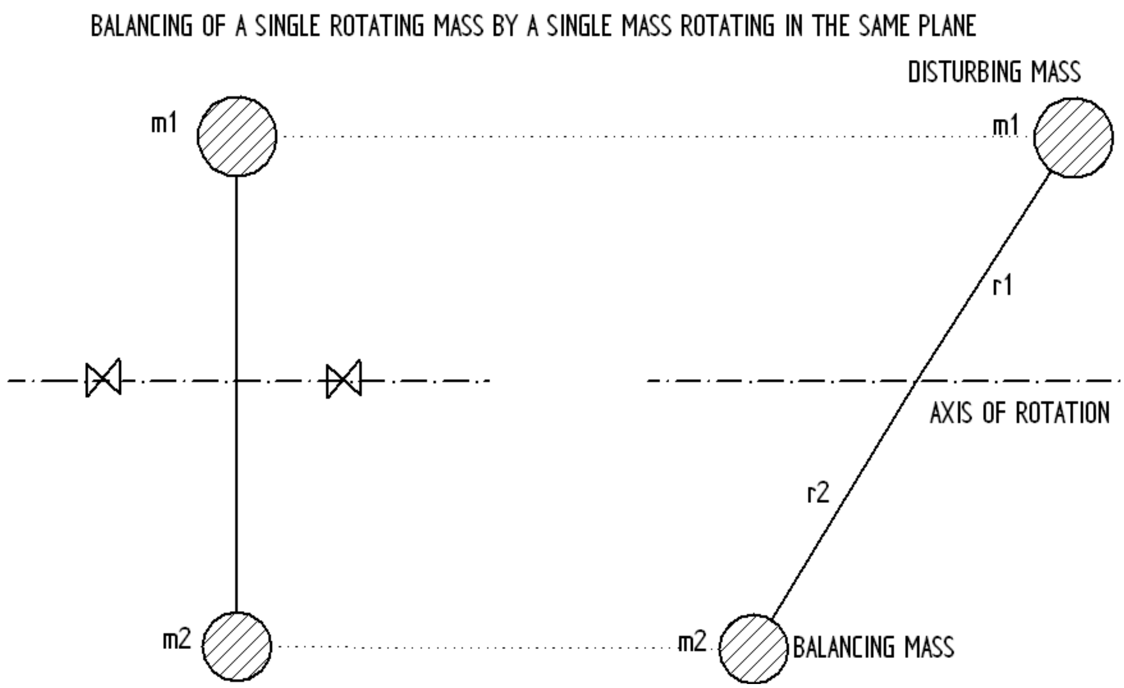
DYNAMIC BALANCING;

When several masses rotate in different planes, the centrifugal forces, in addition to being out of balance, also form couples. A system of rotating masses is in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple.

CASE 1.

BALANCING OF A SINGLE ROTATING MASS ROTATING IN THE SAME PLANE

MASS BY A SINGLE



Consider a disturbing mass m_1 which is attached to a shaft rotating at ω rad/s. Let

r_1 = radius of rotation of the mass m_1
= distance between the axis of rotation of the shaft and the centre of gravity of the mass m_1

The centrifugal force exerted by mass m_1 on the shaft is given by,

$$F_{c1} = m_1 \omega^2 r_1 \quad ---------------------(1)$$

This force acts radially outwards and produces bending moment on the shaft. In order to counteract the effect of this force F_{c1} , a balancing mass m_2 may be attached in the same plane of rotation of the disturbing mass m_1 such that the centrifugal forces due to the two masses are equal and opposite.

Let,

r_2 = radius of rotation of the mass m_2
= distance between the axis of rotation of the shaft and the centre of gravity of the mass m_2

Therefore the centrifugal force due to mass m_2 will be,

$$F_{c2} = m_2 \omega^2 r_2 \text{-----------------------}(2)$$

Equating equations (1) and (2), we get

$$F_{c1} = F_{c2}$$

$$m_1 \omega^2 r_1 = m_2 \omega^2 r_2 \quad \text{or } m_1 r_1 = m_2 r_2 \text{-----------------------}(3)$$

The product $\mathbf{m_2}$ $\mathbf{r_2}$ can be split up in any convenient way. As for as possible the radius of rotation of mass m_2 that is r_2 is generally made large in order to reduce the balancing mass m_2.

CASE 2:

BALANCING OF A SINGLE ROTATING MASS BY TWO MASSES ROTATING IN DIFFERENT PLANES.

There are two possibilities while attaching two balancing masses:

1. The plane of the disturbing mass may be in between the planes of the two balancing masses.

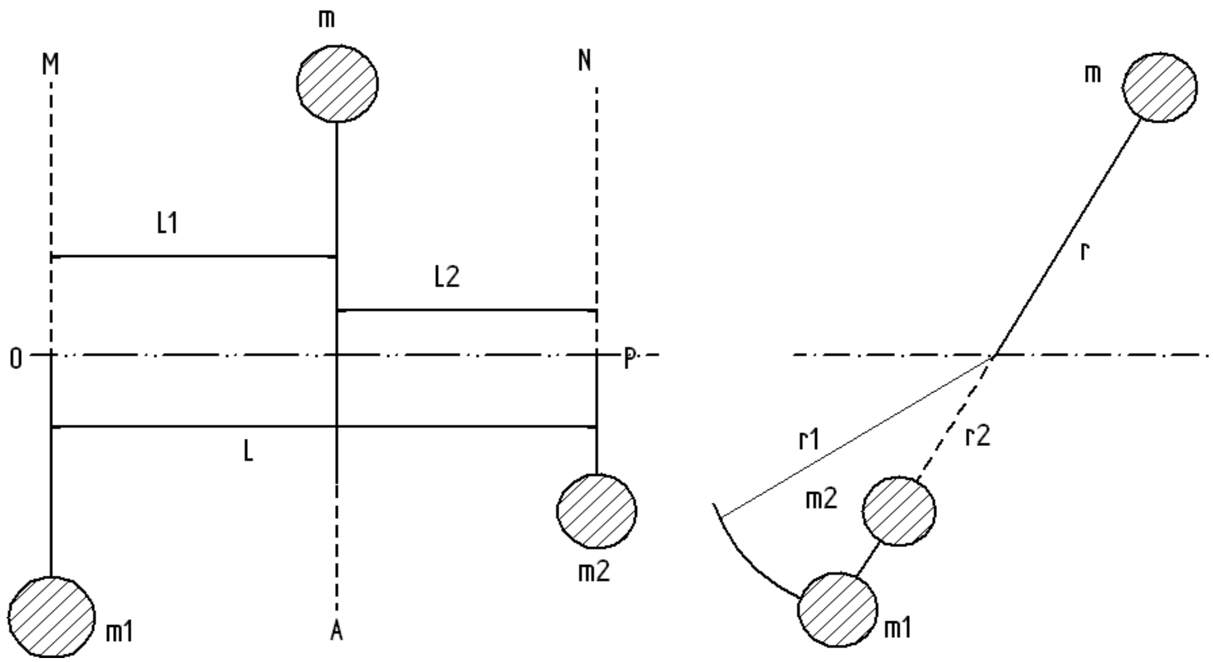
2. The plane of the disturbing mass may be on the left or right side of two planes containing the balancing masses.

In order to balance a single rotating mass by two masses rotating in different planes which are parallel to the plane of rotation of the disturbing mass i) the net dynamic force acting on the shaft must be equal to zero, i.e. the centre of the masses of the system must lie on the axis of rotation and this is the condition for static balancing ii) the net couple due to the dynamic forces acting on the shaft must be equal to zero, i.e. the algebraic sum of the moments about any point in the plane must be zero. The conditions i) and ii) together give dynamic balancing.

CASE 2(I):

THE PLANE OF THE DISTURBING MASS LIES IN BETWEEN THE PLANES OF THE TWO BALANCING MASSES.

The plane of the disturbing mass lies inbetween the planes of the two balancing masses



Consider the disturbing mass m lying in a plane A which is to be balanced by two rotating masses m_1 and m_2 lying in two different planes M and N which are parallel to the plane A as shown.

Let r, r_1 and r_2 be the radii of rotation of the masses in planes A, M and N respectively. Let L_1, L_2 and L be the distance between A and M, A and N, and M and N respectively. Now,
The centrifugal force exerted by the mass m in plane A will be,

$$F_c = m\,\omega^2 r \text{ ------------------- (1)}$$

Similarly,
The centrifugal force exerted by the mass m_1 in plane M will be,

$$F_{c1} = m_1\,\omega^2 r_1 \text{ ------------------- (2)}$$

And the centrifugal force exerted by the mass m_2 in plane N will be,

$$\mathbf{F_{c2} = m_2 \omega^2 r_2 ------------------(3)}$$

For the condition of static balancing,

$$F_c = F_{c1} + F_{c2}$$

$$\text{or } m\omega^2 r = m_1 \omega^2 r_1 + m_2 \omega^2 r_2$$

$$\text{i.e. } m\,r = m_1 r_1 + m_2 r_2 ----------------(4)$$

Now, to determine the magnitude of balancing force in the plane „M" or the dynamic force at the bearing „O" of a shaft, take moments about „ P " which is the point of intersection of the plane N and the axis of rotation.

Therefore,

$$F_{c1} x L = F_c x L_2$$

$$\text{or } m_1 \omega^2 r_1 x L = m\omega^2 r x L_2$$

Therefore,

$$m_1 r_1 L = m r L_2 \quad \text{or } m_1 r_1 = mr \frac{L_2}{L} --------(5)$$

Similarly, in order to find the balancing force in plane „N" or the dynamic force at the bearing „P" of a shaft, take moments about „ O " which is the point of intersection of the plane M and the axis of rotation.

Therefore,

$$F_{c2} x L = F_c x L_1$$

$$\mathbf{or \quad m_2 \omega^2 r_2 x L = m\omega^2 r x L_1}$$

Therefore,

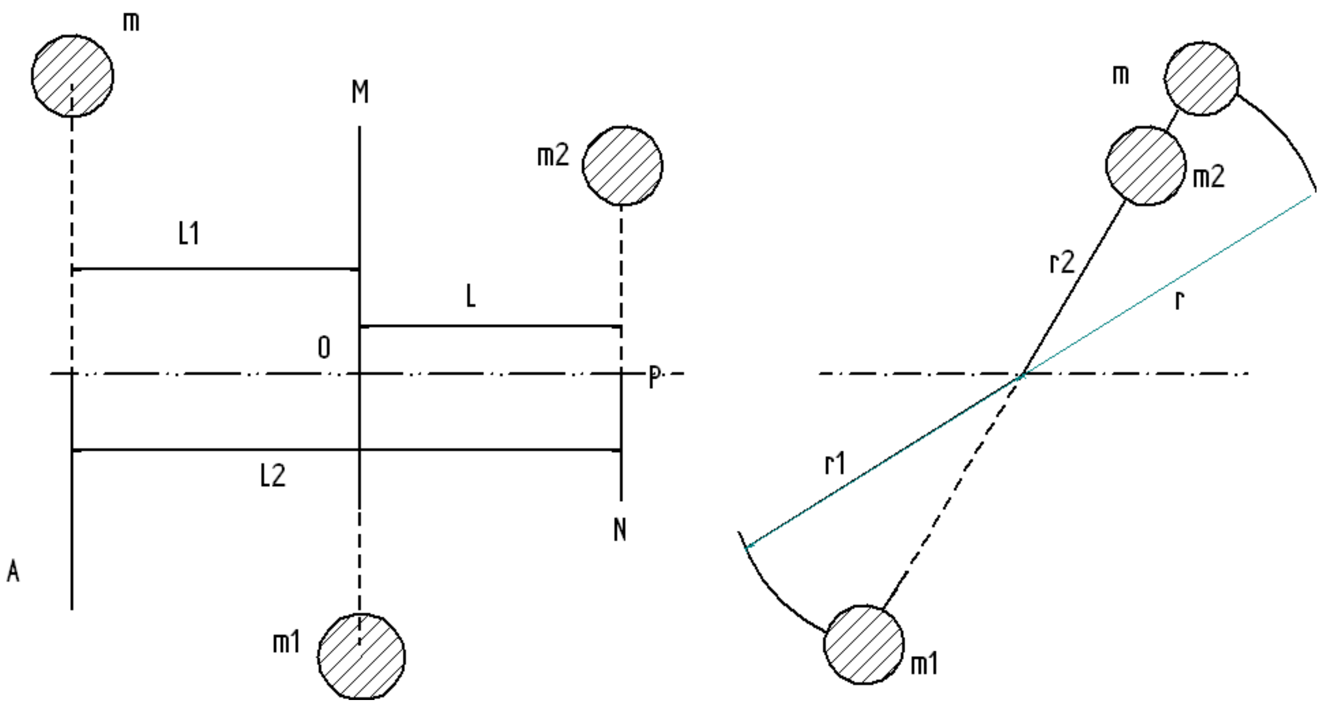
$$m_2 r_2 L = m r L_1 \quad \mathbf{or \ m_2 r_2 = mr \frac{L_1}{L} --------(6)}$$

For dynamic balancing equations (5) or (6) must be satisfied along with equation (4).

CASE 2(II):

WHEN THE PLANE OF THE DISTURBING MASS LIES ON ONE END OF THE TWO PLANES CONTAINING THE BALANCING MASSES.

When the plane of the disturbing mass lies on one end of the planes of the balancing masses



For static balancing,

$$F_{c1} = F_c + F_{c2}$$

$$\text{or } m_1 \omega^2 r_1 = m\omega^2 r + m_2 \omega^2 r_2$$

$$\text{i.e. } m_1 r_1 = m r + m_2 r_2 ----------------(1)$$

For dynamic balance the net dynamic force acting on the shaft and the net couple due to dynamic forces acting on the shaft is equal to zero.
To find the balancing force in the plane „M" or the dynamic force at the bearing „O" of a shaft, take moments about „P". i.e.

$$F_{c1} \times L = F_c \times L_2$$

or $m_1 \omega^2 r_1 \times L = m\omega^2 r \times L_2$ Therefore,

$$m_1 r_1 L = m r L_2 \quad \text{or} \quad m_1 r_1 = m r \frac{L_2}{L} \text{ ---------(2)}$$

Similarly, to find the balancing force in the plane „N" , take moments about „O", i.e.,

$$F_{c2} \times L = F_c \times L_1$$

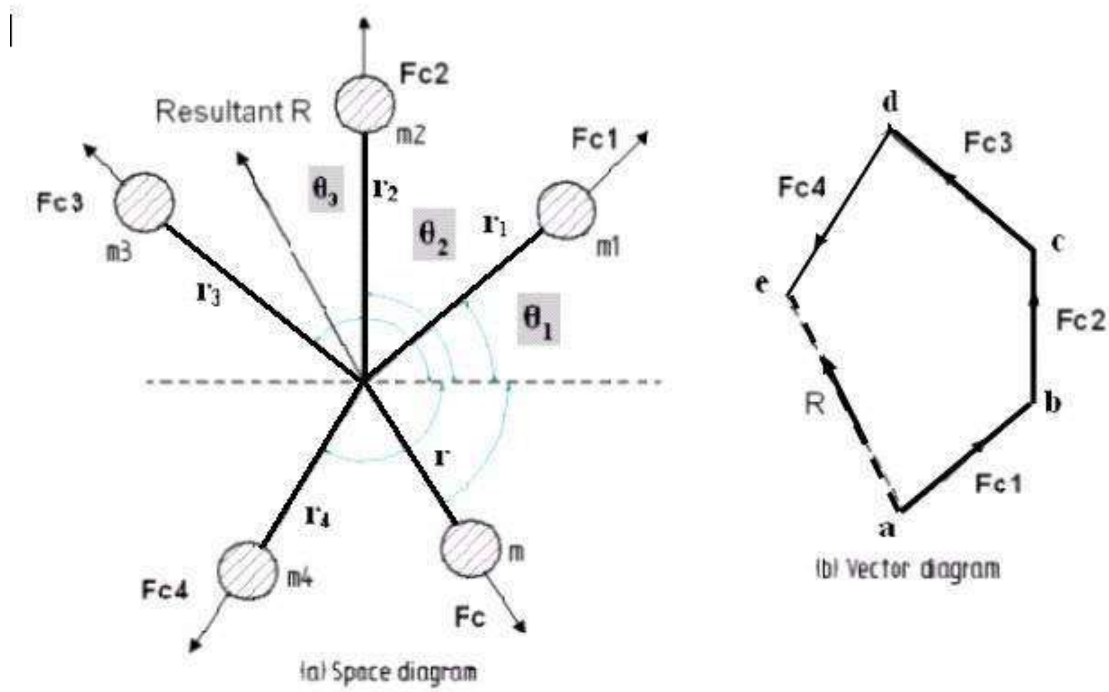
or $m_2 \omega^2 r_2 \times L = m\omega^2 r \times L_1$

Therefore,

$$m_2 r_2 L = m r L_1 \quad \text{or} \quad m_2 r_2 = m r \frac{L_1}{L} \text{ ---------(3)}$$

CASE 3:

BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE



BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE

Consider a rigid rotor revolving with a constant angular velocity ω rad/s. A number of masses say, four are depicted by point masses at different radii in the same transverse plane.

If m_1, m_2, m_3 and m_4 are the masses revolving at radii r_1, r_2, r_3 and r_4 respectively in the same plane.
The centrifugal forces exerted by each of the masses are F_{c1}, F_{c2}, F_{c3} and F_{c4} respectively. Let F be the vector sum of these forces. i.e.

$$F = F_{c1} + F_{c2} + F_{c3} + F_{c4}$$
$$= m_1 \omega^2 r_1 + m_2 \omega^2 r_2 + m_3 \omega^2 r_3 + m_4 \omega^2 r_4 ----------(1)$$

The rotor is said to be statically balanced if the vector sum F is zero. If the vector sum F is not zero, i.e. the rotor is unbalanced, then introduce a counterweight (balance weight) of mass „m" at radius „r" to balance the rotor so that,

$$m_1 \omega^2 r_1 + m_2 \omega^2 r_2 + m_3 \omega^2 r_3 + m_4 \omega^2 r_4 + m \omega^2 r = 0 ---------(2)$$

or

$$m_1 r_1 + m_2 r_2 + m_3 r_3 + m_4 r_4 + m r = 0 -----------------(3)$$

The magnitude of either „m" or „r" may be selected and the other can be calculated. In general, if $\sum \mathbf{m}_i \mathbf{r}_i$ is the vector sum of $\mathbf{m}_1 \mathbf{r}_1$, $\mathbf{m}_2 \mathbf{r}_2$, $\mathbf{m}_3 \mathbf{r}_3$, $\mathbf{m}_4 \mathbf{r}_4$ etc, then,

$$\sum \mathbf{m_i r_i} + \mathbf{mr} = \mathbf{0} --------(4)$$

The above equation can be solved either analytically or graphically.

1. Analytical Method:

Procedure:
Step 1: Find out the centrifugal force or the product of mass and its radius of rotation exerted by each of masses on the rotating shaft, since ω^2 is same for each mass, therefore the magnitude of the centrifugal force for each mass is proportional to the product of the respective mass and its radius of rotation.
Step 2: Resolve these forces into their horizontal and vertical components and find their sums. i.e.,

Sum of the horizontal components

$$= \sum_{i=1}^{n} m_i r_i \cos \theta_i = m_1 r_1 \cos \theta_1 + m_2 r_2 \cos \theta_2 + m_3 r_3 \cos \theta_3 + --------$$

Sum of the vertical components

$$= \sum_{i=1}^{n} m_i r_i \sin \theta_i = m_1 r_1 \sin \theta_1 + m_2 r_2 \sin \theta_2 + m_3 r_3 \sin \theta_3 + --------$$

Step 3: Determine the magnitude of the resultant centrifugal force

$$R = \sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$

Step 4: If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i}$$

Step 5: The balancing force is then equal to the resultant force, but in opposite direction. Step 6: Now find out the magnitude of the balancing mass, such that

R=mr

Where, m = balancing mass and r = its radius of rotation

2. Graphical Method:

Step 1:
Draw the space diagram with the positions of the several masses, as shown.

Step 2:
Find out the centrifugal forces or product of the mass and radius of rotation exerted by each mass.

Step 3:
Now draw the vector diagram with the obtained centrifugal forces or product of the masses and radii of rotation. To draw vector diagram take a suitable scale.
Let ab, bc, cd, de represents the forces F_{c1}, F_{c2}, F_{c3} and F_{c4} on the vector diagram.
Draw „ab‟ parallel to force F_{c1} of the space diagram, at „b‟ draw a line parallel to force F_{c2}. Similarly draw lines cd, de parallel to F_{c3} and F_{c4} respectively.

Step 4:
As per polygon law of forces, the closing side „ae‟ represents the resultant force in magnitude and direction as shown in vector diagram.

Step 5:
The balancing force is then , equal and opposite to the resultant force. Step 6:

Determine the magnitude of the balancing mass (m) at a given radius of rotation (r), such that,

$$F_c = m\omega^2 r$$

or

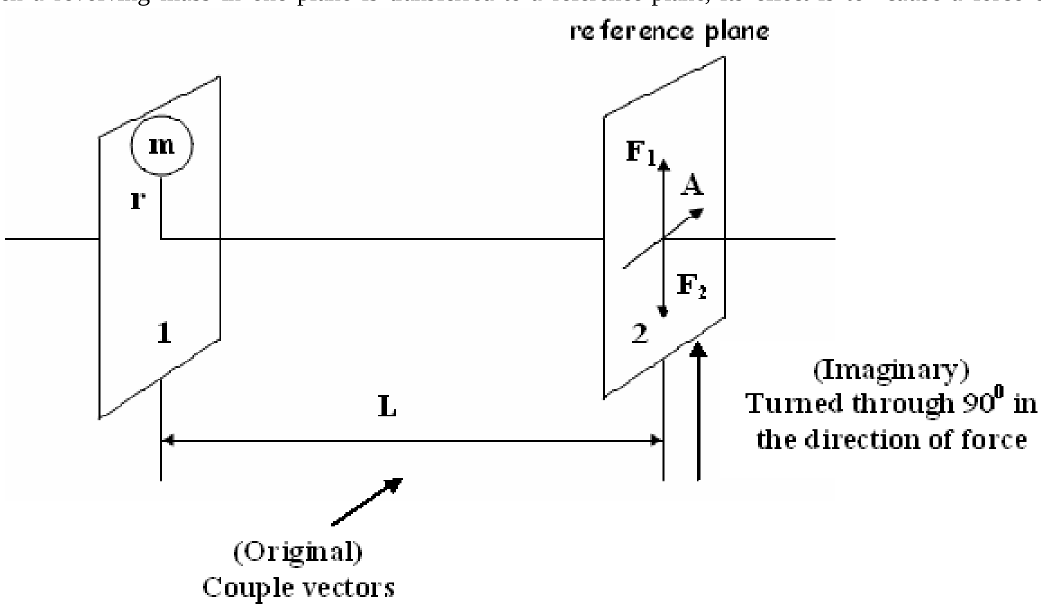
$$mr = \text{resultant of } m_1 r_1, m_2 r_2, m_3 r_3 \text{ and } m_4 r_4$$

CASE 4:

BALANCING OF SEVERAL MASSES ROTATING IN DIFFERENT PLANES

When several masses revolve in different planes, they may be transferred to a reference plane and this reference plane is a plane passing through a point on the axis of rotation and perpendicular to it.

When a revolving mass in one plane is transferred to a reference plane, its effect is to cause a force of same



magnitude to the centrifugal force of the revolving mass to act in the reference plane along with a couple of magnitude equal to the product of the force and the distance between the two planes.

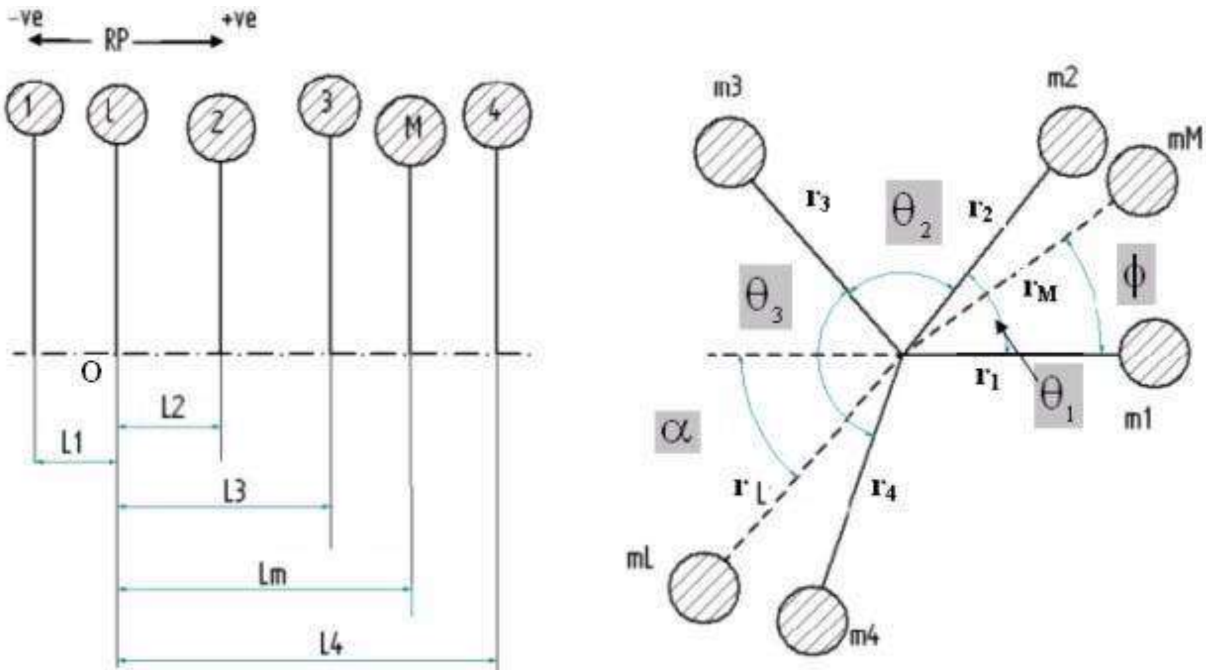
In order to have a complete balance of the several revolving masses in different planes,

1. the forces in the reference plane must balance, i.e., the resultant force must be zero and
2. the couples about the reference plane must balance i.e., the resultant couple must be zero.

A mass placed in the reference plane may satisfy the first condition but the couple balance is satisfied only by two forces of equal magnitude in different planes. Thus, in general, two planes are needed to balance a system of rotating masses.

Example:

Consider four masses m_1, m_2, m_3 and m_4 attached to the rotor at radii r_1, r_2, r_3 and r_4 respectively. The masses m_1, m_2, m_3 and m_4 rotate in planes 1, 2, 3 and 4 respectively.



(a) position of planes of masses (b) Angular position of masses

a) Position of planes of masses

Choose a reference plane at „O‟ so that the distance of the planes 1, 2, 3 and 4 from „O‟ are L_1, L_2 , L_3 and L_4 respectively. The reference plane chosen is plane „L‟. Choose another plane „M‟ between plane 3 and 4 as shown.

Plane „M‟ is at a distance of L_m from the reference plane „L‟. The distances of all the other planes to the left of „L‟ may be taken as negative(-ve) and to the right may be taken as positive (+ve).

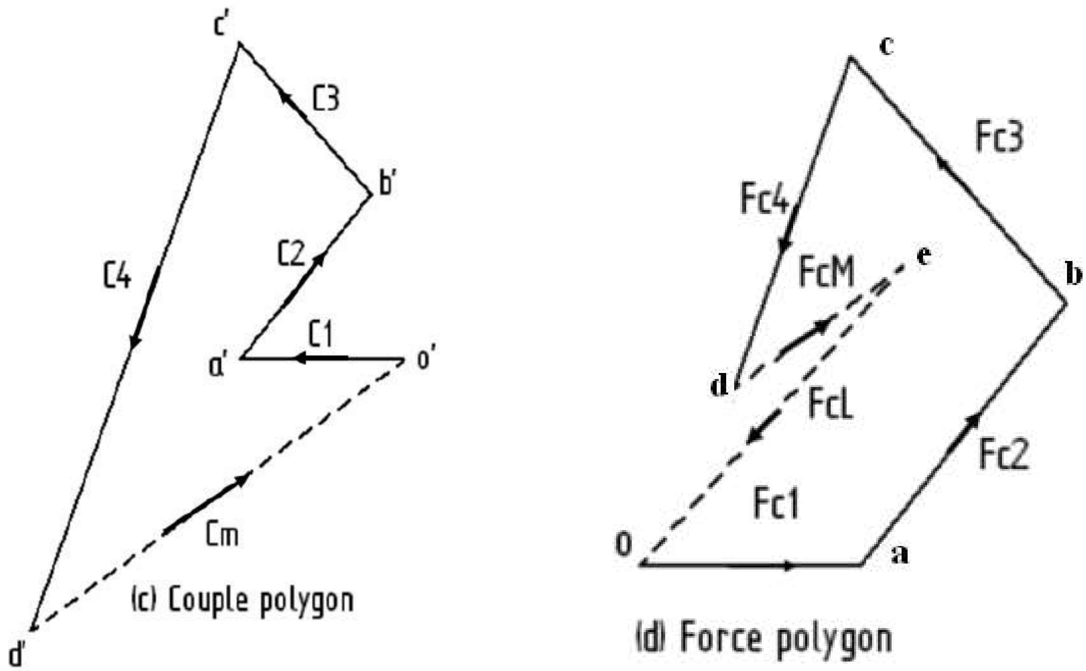
The magnitude of the balancing masses m_L and m_M in planes L and M may be obtained by following the steps given below.

Step 1:

Tabulate the given data as shown after drawing the sketches of position of planes of masses and angular position of masses. The planes are tabulated in the same order in which they occur from left to right.

Plane 1	Mass (m) 2	Radius (r) 3	Centrifugal force/ω^2 (m r) 4	Distance from Ref. plane „L" (L) 5	Couple/ ω^2 (m r L) 6
1	m_1	r_1	$m_1 r_1$	$- L_1$	$- m_1 r_1 L_1$
L	m_L	r_L	$m_L r_L$	0	0
2	m_2	r_2	$m_2 r_2$	L_2	$m_2 r_2 L_2$
3	m_3	r_3	$m_3 r_3$	L_3	$m_3 r_3 L_3$
M	m_M	r_M	$m_M r_M$	L_M	$m_M r_M L_M$
4	m_4	r_4	$m_4 r_4$	L_4	$m_4 r_4 L_4$

Step 2:
Construct the couple polygon first. (The couple polygon can be drawn by taking a convenient scale)
Add the known vectors and considering each vector parallel to the radial line of the mass draw the couple diagram. Then the closing vector will be „m_M r_M L_M".



(c) Couple polygon

(d) Force polygon

The vector d "o" on the couple polygon represents the balanced couple. Since the balanced couple C_M is proportional to m_M r_M L_M , therefore,

$$C_M = m_M \, r_M \, L_M = \text{vector } d' o'$$

$$\text{or} \quad m_M = \frac{\text{vector } d' o'}{r_M \, L_M}$$

From this the value of m_M in the plane M can be determined and the angle of inclination ϕ of this mass may be measured from figure (b).

Step 3:
Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors along with „m_M r_M". The closing vector will be „m_L r_L". This represents the balanced force. Since the balanced force is proportional to „m_L r_L"
,

$$m_L \, r_L = \text{vector } eo$$

$$\text{or} \quad m_L = \frac{\text{vector } eo}{r_L}$$

From this the balancing mass m_L can be obtained in plane „L" and the angle of inclination of this mass with the horizontal may be measured from figure (b).

Problems and solutions

Problem 1.

Four masses A, B, C and D are attached to a shaft and revolve in the same plane. The masses are 12 kg, 10 kg, 18 kg and 15 kg respectively and their radii of rotations are 40 mm, 50 mm, 60 mm and 30 mm. The angular position of the masses B, C and D are 60^0 , 135^0 and 270^0 from mass A. Find the magnitude and position of the balancing mass at a radius of 100 mm.

Solution:
Given:

Mass(m) kg	Radius(r) m	Centrifugal force/ω^2 (m r) kg-m	Angle(θ)
m_A = 12 kg (reference mass)	r_A = 0.04 m	$m_A r_A$ = 0.48 kg-m	$\theta_A = 0^0$
m_B = 10 kg	r_B = 0.05 m	$m_B r_B$ = 0.50 kg-m	$\theta_B = 60^0$
m_C = 18 kg	r_C = 0.06 m	$m_C r_C$ = 1.08 kg-m	$\theta_C = 135^0$
m_D = 15 kg	r_D = 0.03 m	$m_D r_D$ = 0.45 kg-m	$\theta_D = 270^0$

To determine the balancing mass „m" at a radius of r = 0.1 m.

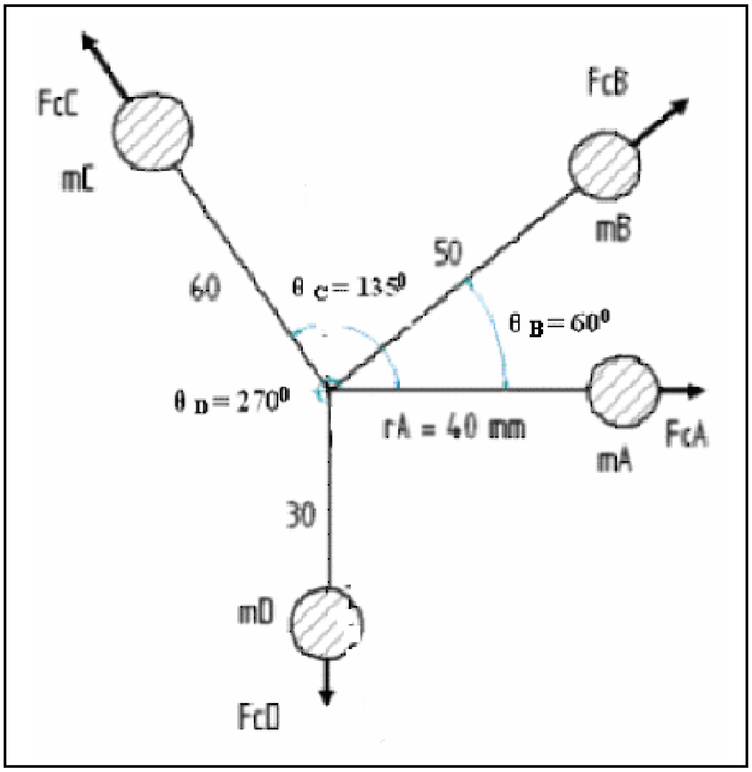
The problem can be solved by either analytical or graphical method.

Analytical Method:

Step 1:

Draw the space diagram or angular position of the masses. Since all the angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta = 0^0$.

A



Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product „mr" can be calculated and tabulated.

Step 2:

Resolve the centrifugal forces horizontally and vertically and find their sum. Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ horizontally and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \cos\theta_i = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D$$

$$= 0.48 \text{ x } \cos 0^0 + 0.50 \text{ x } \cos 60^0 + 1.08 \text{ x } \cos 135^0 + 0.45 \text{ x } \cos 270^0$$

$$= 0.48 + 0.25 + (-0.764) + 0 = -0.034 \text{ kg-m} \text{ ---------- } (1)$$

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ vertically and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \sin\theta_i = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D$$

$$= 0.48 \times \sin 0^0 + 0.50 \times \sin 60^0 + 1.08 \times \sin 135^0 + 0.45 \times \sin 270^0$$

$$= 0 + 0.433 + 0.764 + (-0.45) = 0.747 \text{ kg-m} \quad ----------(2)$$

Step 3:

Determine the magnitude of the resultant centrifugal force

$$R = \sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$

$$= \sqrt{(-0.034)^2 + (0.747)^2} = 0.748 \text{kg} - \text{m}$$

Step 4:

The balancing force is then equal to the resultant force, but in opposite direction. Now find out the magnitude of the balancing mass, such that

$$R = mr = 0.748 \text{kg} - \text{m}$$

$$\text{Therefore,} \quad m = \frac{R}{r} = \frac{0.748}{0.1} = 7.48 \text{ kg Ans}$$

Where, m = balancing mass and r = its radius of rotation

Step 5:

Determine the position of the balancing mass „m".

If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i} = \frac{0.747}{-0.034} = -21.97$$

$$\text{and } \theta = -87.4^0 \text{ or } 92.6^0$$

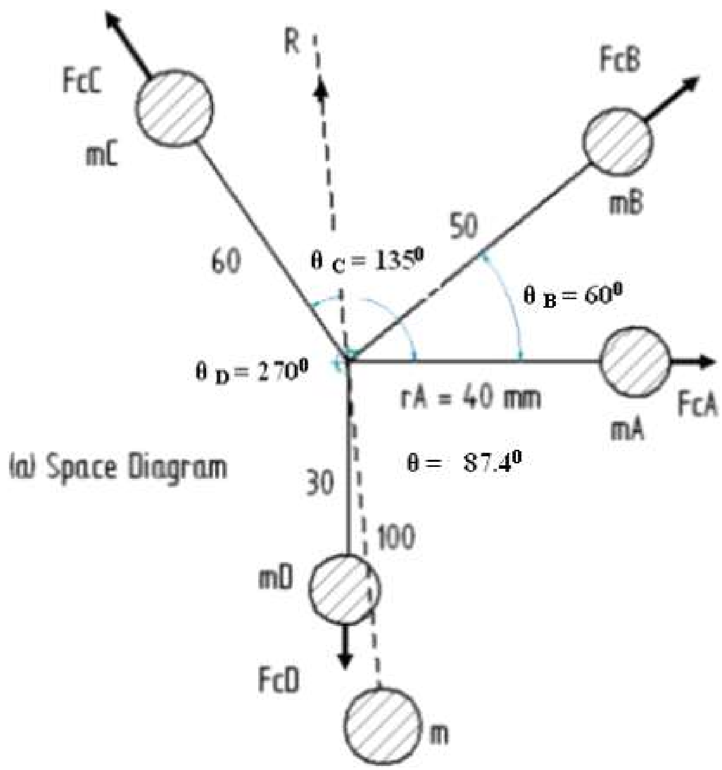
Remember ALL STUDENTS TAKE COPY i.e. in first quadrant all angles **(sin θ, cosθ and tanθ)** are positive, in second quadrant only **sinθ** is positive, in third quadrant only **tanθ** is positive and in fourth quadrant only $\cos\theta$ is positive.

Since numerator is positive and denominator is negative, the resultant force makes with the horizontal, an angle (measured in the counter clockwise direction)

$$\theta = 92.6^{0}$$

The balancing force is then equal to the resultant force, but in opposite direction.

The balancing mass „m" lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal clockwise is, $\theta_M = 87.4^0$ angle measured in the direction.



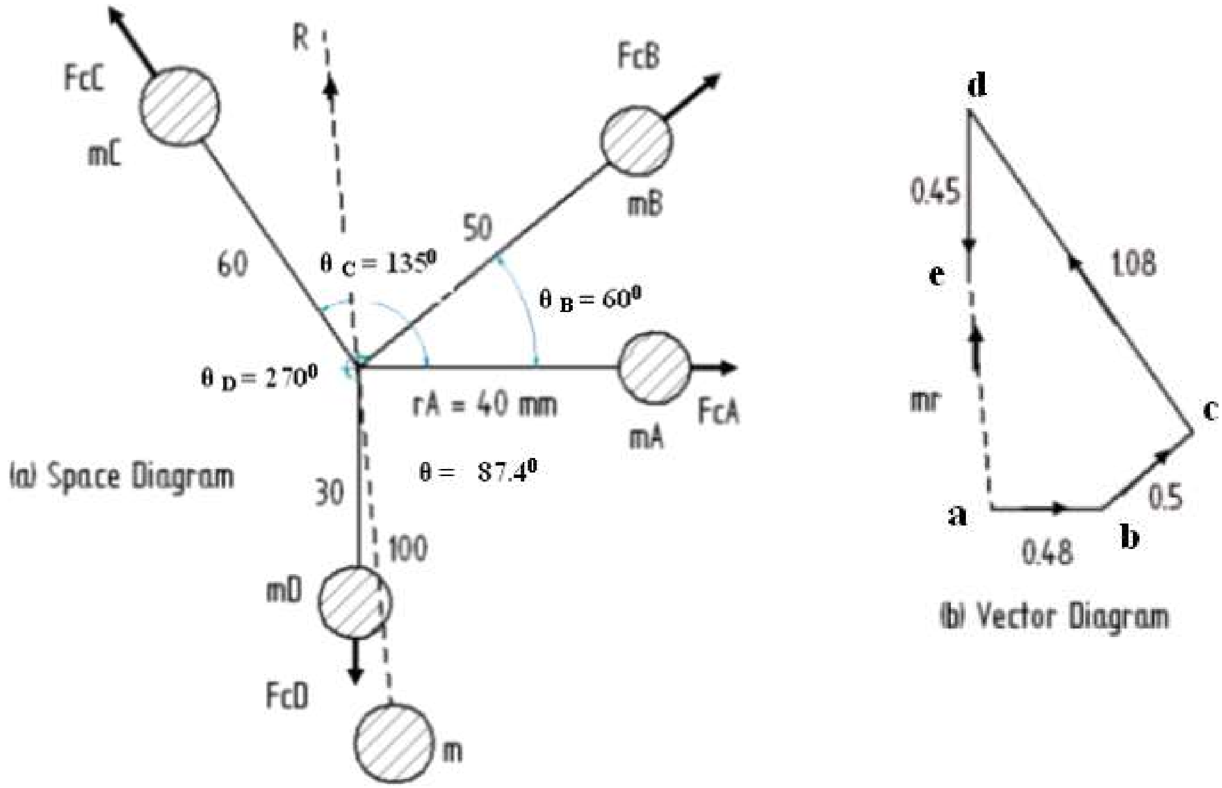
Graphical Method:

Step 1:

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product „mr" can be calculated and tabulated.

Draw the space diagram or angular position of the masses taking the actual angles(Since all angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta = 0^0$).

A



Step 2:

Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors as follows.

Draw a line „ab‟ parallel to force F_{CA} (or the product $m_A r_A$ to a proper scale) of the space diagram. At „b‟ draw a line „bc‟ parallel to F_{CB} (or the product $m_B r_B$). Similarly draw lines „cd‟, „de‟ parallel to F_{CC} (or the product $m_C r_C$) and F_{CD} (or the product $m_D r_D$) respectively. The closing side „ae‟ represents the resultant force „R‟ in magnitude and direction as shown on the vector diagram.

Step 3:

The balancing force is then equal to the resultant force, but in opposite direction.

$$R = mr$$

$$\text{Therefore, } m = \frac{R}{r} = 7.48 \text{ kg Ans}$$

The balancing mass „m‟ lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta_M = 87.4^0$ angle measured in the clockwise direction.

Problem 2:

The four masses A, B, C and D are 100 kg, 150 kg, 120 kg and 130 kg attached to a shaft and revolve in the same plane. The corresponding radii of rotations are 22.5 cm, 17.5 cm, 25 cm and 30 cm and the angles measured from A are 45^0, 120^0 and 255^0. Find the position and magnitude of the balancing mass, if the radius of rotation is 60 cm.

Solution:

Analytical Method:

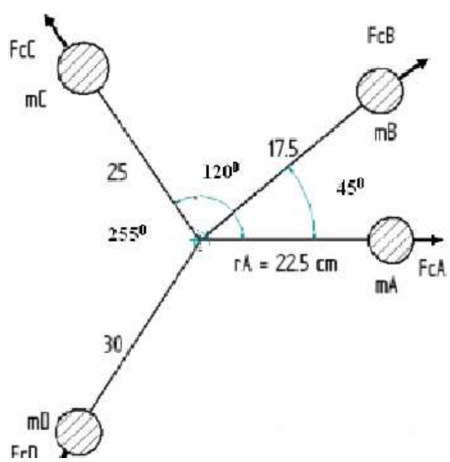
Given:

Mass(m) kg	Radius(r) m	Centrifugal force/ω^2 (m r) kg-m	Angle(θ)
m_A = 100 kg (reference mass)	r_A = 0.225 m	$m_A r_A$ = 22.5 kg-m	$\theta_A = 0^0$
m_B = 150 kg	r_B = 0.175 m	$m_B r_B$ = 26.25 kg-m	$\theta_B = 45^0$
m_C = 120 kg	r_C = 0.250 m	$m_C r_C$ = 30 kg-m	$\theta_C = 120^0$
m_D = 130 kg	r_D = 0.300 m	$m_D r_D$ = 39 kg-m	$\theta_D = 255^0$
m =?	r = 0.60		θ = **?**

Step 1:

Draw the space diagram or angular position of the masses. Since all the angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$.

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product „mr" can be calculated and tabulated.



Step 2:

Resolve the centrifugal forces horizontally and vertically and find their sum. Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ horizontally and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \cos\theta_i = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D$$

$$= 22.5 \text{ x } \cos 0^0 + 26.25 \text{ x } \cos 45^0 + 30 \text{ x } \cos 120^0 + 39 \text{ x } \cos 255^0$$

$$= 22.5 + 18.56 + (-15) + (-10.1) = 15.97 \text{ kg} - \text{m} \qquad ----------(1)$$

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ vertically and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \sin\theta_i = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D$$

$$= 22.5 \text{ x } \sin 0^0 + 26.25 \text{ x } \sin 45^0 + 30 \text{ x } \sin 120^0 + 39 \text{ x } \sin 255^0$$

$$= 0 + 18.56 + 25.98 + (-37.67) = 6.87 \text{ kg} - \text{m} \qquad ----------(2)$$

Step 3:

Determine the magnitude of the resultant centrifugal force

$$R = \sqrt{\left[\sum_{i=1}^{n} m_i r_i \cos\theta_i\right]^2 + \left[\sum_{i=1}^{n} m_i r_i \sin\theta_i\right]^2}$$

$$= \sqrt{(15.97)^2 + (6.87)^2} = 17.39 \text{ kg} - \text{m}$$

Step 4:

The balancing force is then equal to the resultant force, but in opposite direction. Now find out the magnitude of the balancing mass, such that

$$R = m\,r = 17.39 \text{ kg} - \text{m}$$

$$\text{Therefore,} \quad m = \frac{R}{r} = \frac{17.39}{0.60} = 28.98 \text{ kg Ans}$$

Where, m = balancing mass and r = its radius of rotation

Step 5:

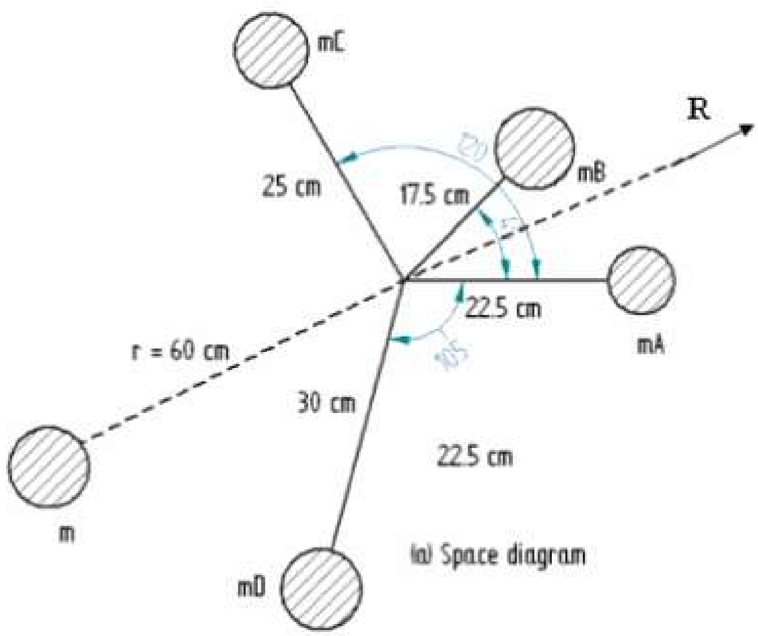
Determine the position of the balancing mass „m".

If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i} = \frac{6.87}{15.97} = 0.4302$$

$$\text{and } \theta = 23.28^0$$

The balancing mass „m" lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta = 203.28^0$ angle measured in the counter clockwise direction.



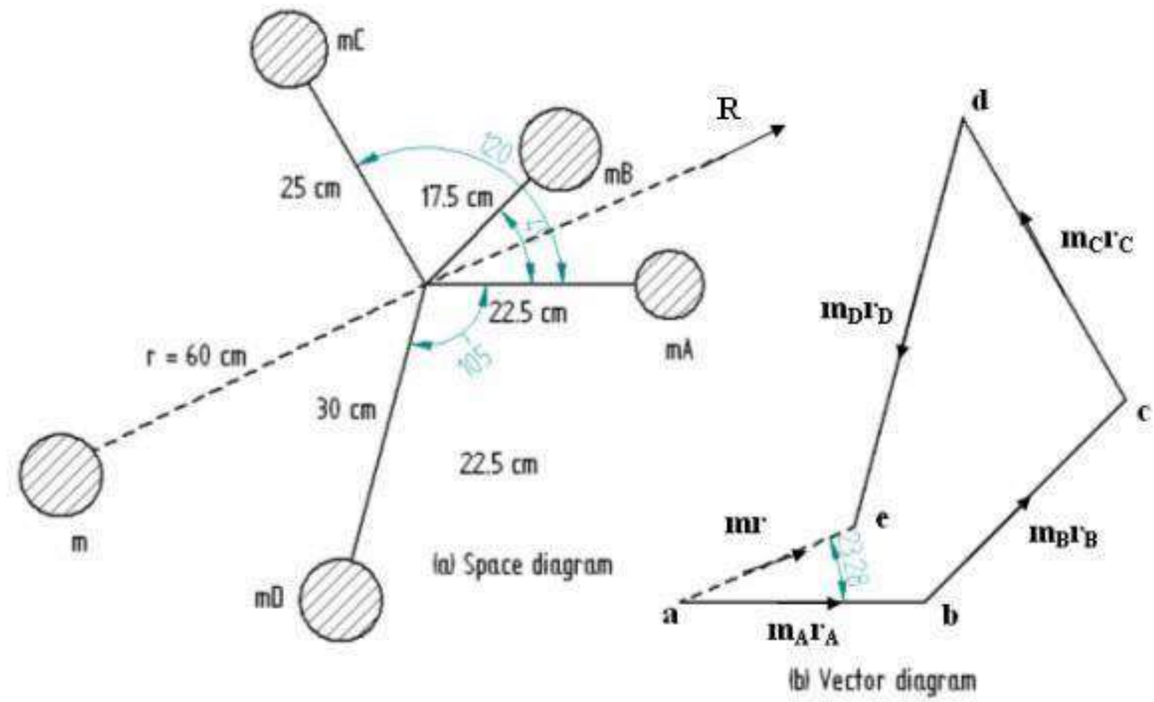
Graphical Method:

Step 1:

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product „mr" can be calculated and tabulated.

Step 2:

Draw the space diagram or angular position of the masses taking the actual angles (Since all angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta = 0^0$).



Draw a line „ab" parallel to force F_{CA} (or the product $m_A r_A$ to a proper scale) of the space diagram. At „b" draw a line „bc" parallel to F_{CB} (or the product $m_B r_B$). Similarly draw lines „cd", „de" parallel to F_{CC} (or the product $m_C r_C$) and F_{CD} (or the product $m_D r_D$) respectively. The closing side „ae" represents the resultant force „R" in magnitude and direction as shown on the vector diagram.

Step 4:

The balancing force is then equal to the resultant force, but in opposite direction.

$$R = mr$$

$$\text{Therefore, } m = \frac{R}{r} = 29 \text{ kg Ans}$$

The balancing mass „m" lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta = 203^0$ angle measured in the counter clockwise direction.

Problem 3:

A rotor has the following properties.

Mass	magnitude	Radius	Angle	Axial distance from first mass
1	9 kg	100 mm	$\theta_A = 0^0$	-
2	7 kg	120 mm	$\theta_B = 60^0$	160 mm
3	8 kg	140 mm	$\theta_C = 135^0$	320 mm
4	6 kg	120 mm	$\theta_D = 270^0$	560 mm

If the shaft is balanced by two counter masses located at 100 mm radii and revolving in planes midway of planes 1 and 2, and midway of 3 and 4, determine the magnitude of the masses and their respective angular positions.

Solution:

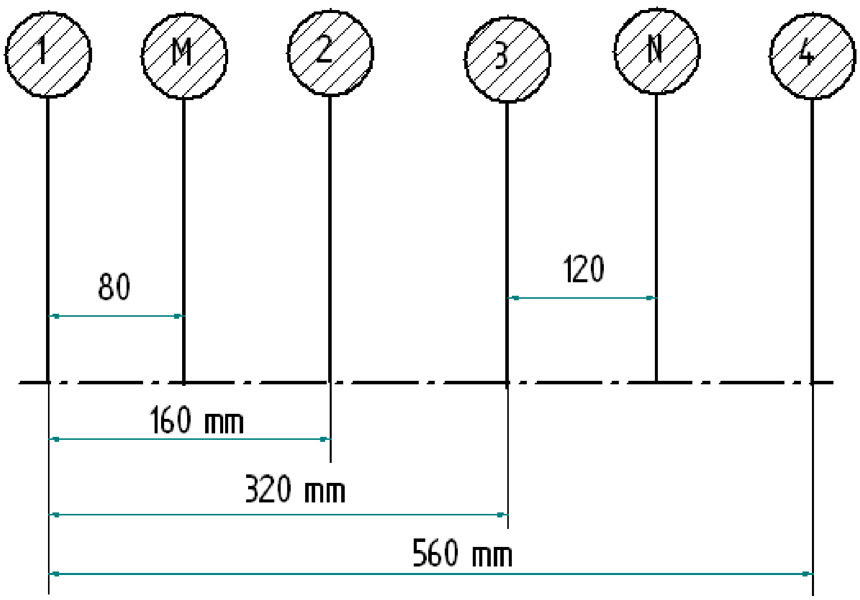
Analytical Method:

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „M" m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
1	9.0	0.10	$m_1 r_1 = 0.9$	-0.08	-0.072	0^0
M	m_M = ?	0.10	$m_M r_M = 0.1\, m_M$	0	0	θ_M=**?**
2	7.0	0.12	$m_2 r_2 = 0.84$	0.08	0.0672	60^0
3	8.0	0.14	$m_3 r_3 = 1.12$	0.24	0.2688	135^0
N	m_N = ?	0.10	$m_N r_N = 0.1\, m_N$	0.36	$m_N r_N l_N = 0.036\, m_N$	θ_N=**?**
4	6.0	0.12	$m_4 r_4 = 0.72$	0.48	0.3456	270^0

For dynamic balancing the conditions required are,

$$\sum mr + m_M r_M + m_N r_N = 0 \quad \text{------------ (I)} \quad \text{for force balance}$$

$$\sum mrl + m_N r_N l_N = 0 \quad \text{----------------- (II)} \quad \text{for couple balance}$$



(a) Position of planes of masses

Step 1:

Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta + m_N r_N l_N \cos\theta_N = 0$$

On substitution we get

$$-0.072\cos 0^0 + 0.0672\cos 60^0 + 0.2688\cos 135^0$$

$$+0.3456\cos 270^0 + 0.036\, m_N \cos\theta_N = 0$$

i.e. $0.036\, m_N \cos\theta_N = 0.2285 ----- (1)$

Sum of the vertical components gives,

$$\sum mrl\sin\theta + m_N r_N l_N \sin\theta_N = 0$$

On substitution we get

$$-0.072\sin 0^0 + 0.0672\sin 60^0 + 0.2688\sin 135^0$$

$$+0.3456\sin 270^0 + 0.036\, m_N \sin\theta_N = 0$$

i.e. $0.036\, m_N \sin\theta_N = 0.09733 ----- (2)$

Squaring and adding (1) and (2), we get

$$m_N r_N l = \sqrt{(0.2285)^2 + (0.09733)^2}$$

$$\text{i.e., } 0.036 m_N = 0.2484$$

$$\text{Therefore, } \quad m_N = \frac{0.2484}{0.036} = 6.9\text{kg Ans}$$

Dividing (2) by (1), we get

$$\tan\theta_N = \frac{0.09733}{0.2285} \quad \text{and} \quad \theta_N = 23.07^0$$

Step 2:

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_M r_M \cos\theta_M + m_N r_N \cos\theta_N = 0$$

On substitution we get

$$0.9\cos 0^0 + 0.84\cos 60^0 + 1.12\cos 135^0 + 0.72\cos 270^0$$
$$+ m_M r_M \cos\theta_M + 0.1x6.9x\cos 23.07^0 = 0$$

$$\text{i.e. } m_M r_M \cos\theta_M = -1.1629 \text{ -----}(3)$$

Sum of the vertical components gives,

$$\sum mr\sin\theta + m_M r_M \sin\theta_M + m_N r_N \sin\theta_N = 0$$

On substitution we get

$$0.9\sin 0^0 + 0.84\sin 60^0 + 1.12\sin 135^0 + 0.72\sin 270^0$$
$$+ m_M r_M \sin\theta_M + 0.1x6.9x\sin 23.07^0 = 0$$
$$\text{i.e. } m_M r_M \sin\theta_M = -1.0698 \text{ -----}(4)$$

Squaring and adding (3) and (4), we get

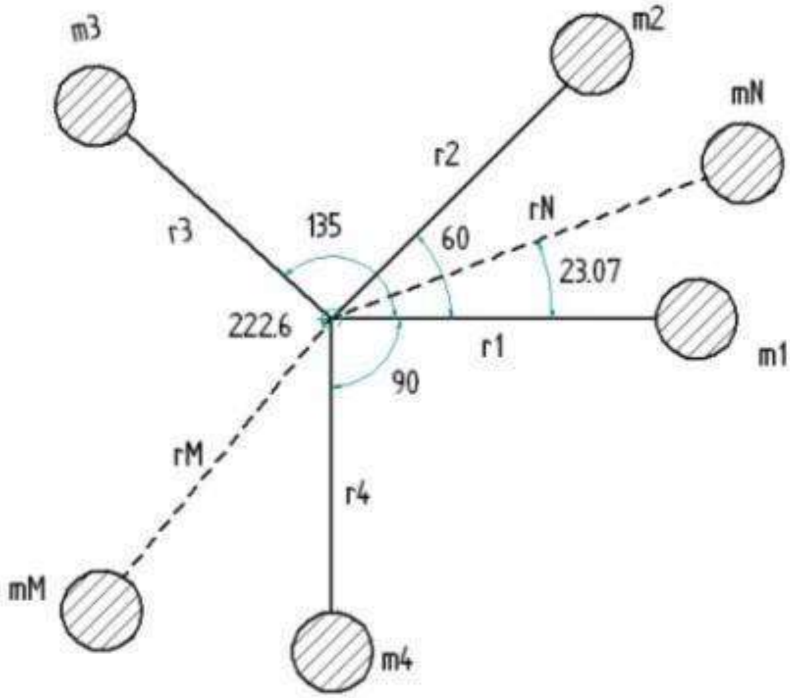
$$m_M r_M = \sqrt{(-1.1629)^2 + (-1.0698)^2}$$

$$\text{i.e., } 0.1 m_M = 1.580$$

$$\text{Therefore, } \quad m_M = \frac{1.580}{0.1} = 15.8 \text{ kg Ans}$$

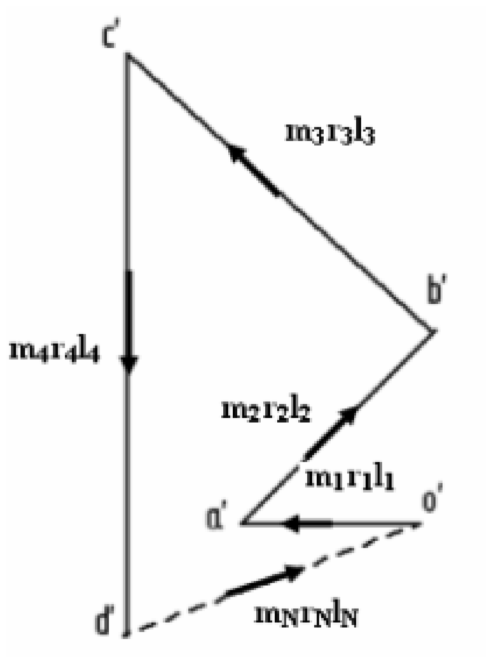
Dividing (4) by (3), we get

$$\tan\theta_M = \frac{-1.0698}{-1.1629} \quad \text{and } \theta_M = 222.61^0 \text{ Ans}$$

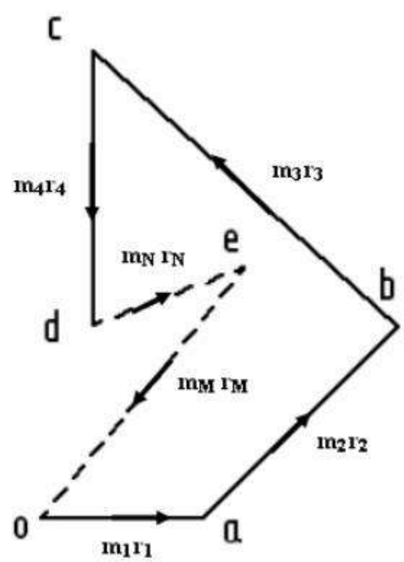


(b) Angular position of masses

Graphical Solution:



(c) Couple polygon



(d) Force polygon

Problem 4:
The system has the following data.

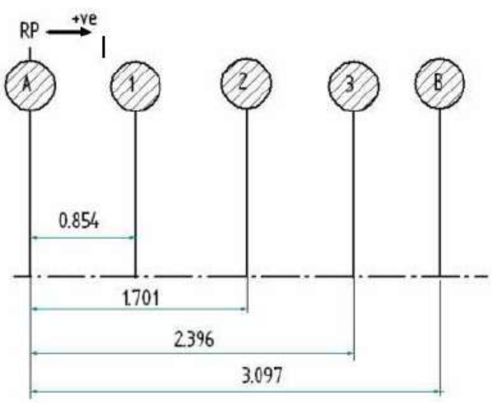
$\mathbf{m_1}$ =1.2 **kg**	$\mathbf{r_1}$ = 1.135 **m@** $\angle$ 113.4^0
$\mathbf{m_1}$ =1.8 **kg**	$\mathbf{r_2}$ = 0.822 **m@** $\angle$ 48.8^0
$\mathbf{m_1}$ = 2.4 **kg**	$\mathbf{r_3}$ = 1.04 **m@** $\angle$ 251.4^0

The distances of planes in metres from plane A are:

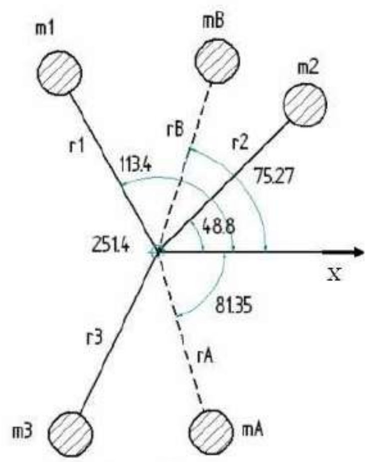
$$l_1 = 0.854\ ,\ l_2 = 1.701, l_3 = 2.396\ , l_B = 3.097$$

Find the mass-radius products and their angular locations needed to dynamically balance the system using the correction planes A and B.

Solution: Analytical Method



(a) Position of planes of masses



(b) Angular position of masses

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „A" m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A	m_A	r_A	$m_A\ r_A$ =?	0	0	θ_A=**?**
1	1.2	1.135	1.362	0.854	1.163148	113.4^0
2	1.8	0.822	1.4796	1.701	2.5168	48.8^0
3	2.4	1.04	2.496	2.396	5.9804	251.4^0
B	m_B	r_B	$m_B\ r_B$ =?	3.097	3.097 $m_B\ r_B$	θ_B=**?**

Step 1:

Resolve the couples into their horizontal and vertical components and find their sums. Sum of the horizontal components gives,

$$\sum mrl\cos\theta + m_B r_B l_B \cos\theta_B = 0$$

On substitution we get

$$1.163148\cos 113.4^0 + 2.5168\cos 48.8^0 + 5.9804\cos 251.4^0$$

$$+3.097\, m_B r_B \cos\theta_B = 0$$

$$\text{i.e. } m_B r_B \cos\theta_B = \frac{0.71166}{3.097} \quad -----(1)$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta + m_B r_B l_B \sin\theta_B = 0$$

On substitution we get

$$1.163148\sin 113.4^0 + 2.5168\sin 48.8^0 + 5.9804\sin 251.4^0$$

$$+3.097\, m_B r_B \sin\theta_B = 0$$

$$\text{i.e. } m_B r_B \sin\theta_B = \frac{2.7069}{3.097} \quad -----(2)$$

Squaring and adding (1) and (2), we get

$$m_B r_B = \sqrt{\left(\frac{0.71166}{3.097}\right)^2 + \left(\frac{2.7069}{3.097}\right)^2}$$

$$= 0.9037\ \text{kg} - \text{m}$$

Dividing (2) by (1), we get

$$\tan\theta_B = \frac{2.7069}{0.71166} \quad \text{and } \theta_B = 75.27^0\ \text{Ans}$$

Step 2:

Resolve the forces into their horizontal and vertical components and find their sums. Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_A r_A \cos\theta_A + m_B r_B \cos\theta_B = 0$$

On substitution we get

$$1.362\cos 113.4^0 + 1.4796\cos 48.8^0 + 2.496\cos 251.4^0$$

$$+ m_A r_A \cos\theta_A + 0.9037\cos 75.27^0 = 0$$

Therefore

$$m_A r_A \cos\theta_A = 0.13266 ---------- (3)$$

Sum of the vertical components gives,

$$\sum mr\sin\theta + m_A r_A \sin\theta_A + m_B r_B \sin\theta_B = 0$$

On substitution we get

$$1.362\sin 113.4^0 + 1.4796\sin 48.8^0 + 2.496\sin 251.4^0$$

$$+ m_A r_A \sin\theta_A + 0.9037\sin 75.27^0 = 0$$

Therefore

$$m_A r_A \sin\theta_A = -0.87162 ---------- (4)$$

Squaring and adding (3) and (4), we get

$$m_A r_A = \sqrt{(0.13266)^2 + (-0.87162)^2}$$

$$= 0.8817\ \text{kg} - \text{m}$$

Dividing (4) by (3), we get

$$\tan\theta_A = \frac{-0.87162}{0.13266} \quad \text{and } \theta_A = -81.35^0 \text{ Ans}$$

Problem 5:

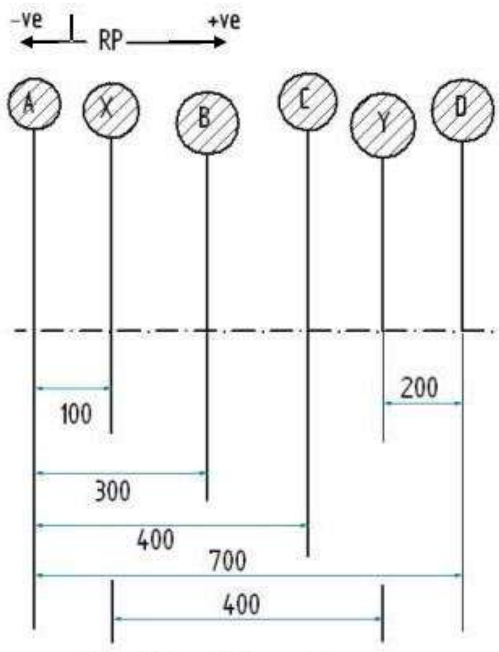
A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 80 mm, 70 mm, 60 mm and 80 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B 45^0, B to C 70^0 and C to D 120^0. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions.

Graphical solution:

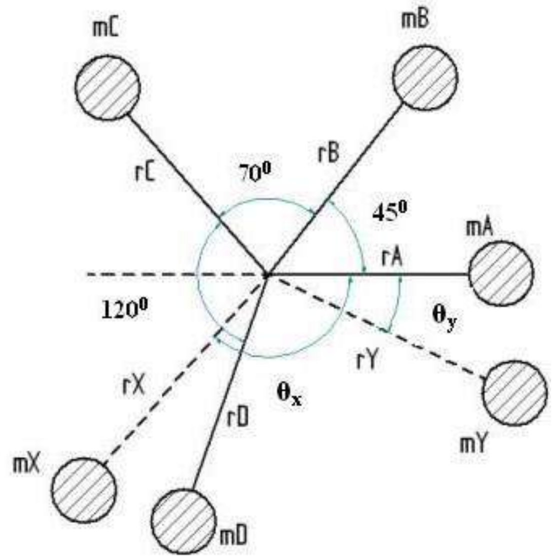
Let, m_X be the balancing mass placed in plane X and m_Y be the balancing mass placed in plane Y which are to be determined.

Step 1:

Draw the position of the planes as shown in figure (a).



(a) position of planes of masses



(b) Angular position of masses

Let X be the reference plane (R.P.). The distances of the planes to the right of the plane X are taken as positive (+ve) and the distances of planes to the left of X plane are taken as negative(-ve). The data may be tabulated as shown

Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product „m r‟ can be calculated and tabulated. Similarly the magnitude of the couples are proportional to the product of the mass , its radius and the axial distance from the reference plane, the product „m r l‟ can be calculated and tabulated as shown.

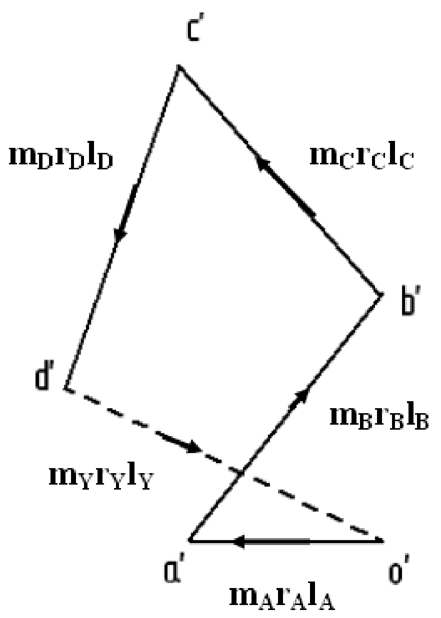
Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „X‟ m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A	200	0.08	m_A r_A = 16	-0.10	-1.60	-
X	m_X =?	0.10	m_X r_X = 0.1 m_X	0	0	θ_X=**?**
B	300	0.07	m_B r_B = 21	0.20	4.20	A to B 45^0
C	400	0.06	m_C r_C = 24	0.30	7.20	B to C 70^0
Y	m_Y =?	0.10	m_Y r_Y = 0.1 m_Y	0.40	m_Y r_Y l_Y = 0.04 m_Y	θ_Y=**?**
D	200	0.08	m_D r_D = 16	0.60	9.60	C to D 120^0

Step 2:

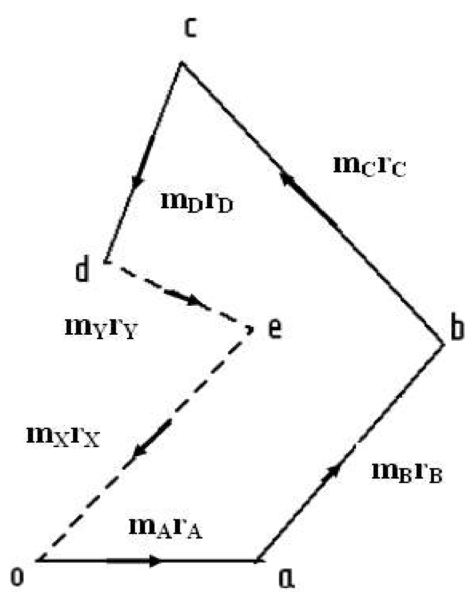
Assuming the mass A as horizontal draw the sketch of angular position of masses as shown in figure (b).

Step 3:

Draw the couple polygon to some suitable scale by taking the values of „m r l‟ (column no. 6) of the table as shown in figure (c).



(c) Couple polygon



(d) Force polygon

Draw line o‟a‟ parallel to the radial line of mass m_A.
At a‟ draw line a‟b‟ parallel to radial line of mass m_B.
Similarly, draw lines b‟c‟, c‟d‟ parallel to radial lines of masses m_C and m_D respectively. Now, join d‟ to o‟ which gives the balanced couple.

We get,

$$0.04\, m_Y = \text{vector } d'o' = 7.3 \text{ kg} - m^2$$

$$\text{or} \quad m_Y = 182.5 \text{ kg Ans}$$

Step 4:

To find the angular position of the mass m_Y draw a line om_Y in figure (b) parallel to d"o" of the couple polygon.

By measurement we get $\theta_Y = 12^0$ in the clockwise direction from m_A.

Step 5:

Now draw the force polygon by considering the values of „m r" (column no. 4) of the table as shown in figure (d).
Follow the similar procedure of step 3. The closing side of the force polygon i.e. „e o" represents the balanced force.

$$m_x r_X = \text{vector } eo = 35.5 \text{ kg} - m$$

$$\text{or} \quad m_x = 355 \text{ kg Ans}$$

Step 6:

The angular position of m_X is determined by drawing a line om_X parallel to the line „e o" of the force polygon in figure (b). From figure (b) we get,
$\theta_X = 145^0$, measured clockwise from m_A. Ans

Problem 6:

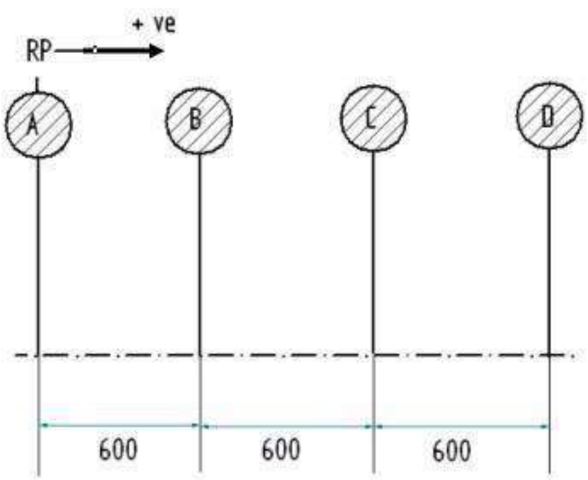
A, B, C and D are four masses carried by a rotating shaft at radii 100 mm, 125 mm, 200 mm and 150 mm respectively. The planes in which the masses revolve are spaced 600 mm apart and the mass of B, C and D are 10 kg, 5 kg and 4 kg respectively. Find the required mass A and relative angular settings of the four masses so that the shaft shall be in complete balance.
Solution:

Graphical Method:

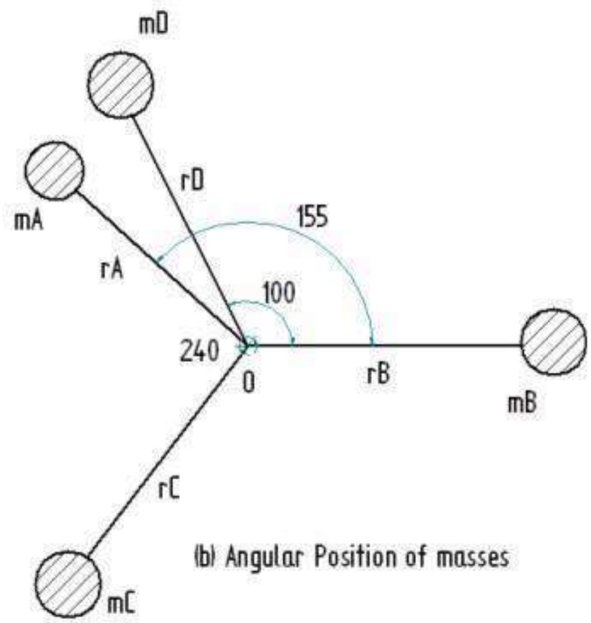
Step 1:

Let, m_A be the balancing mass placed in plane A which is to be determined along with the relative angular settings of the four masses.
Let A be the reference plane (R.P.). Assume the mass B as horizontal
Draw the sketch of angular position of mass m_B (line om_B) as shown in figure (b). The data may be tabulated as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „A" m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	m_A= ?	0.1	$m_A r_A$ = 0.1 m_A	0	0	θ_A=?
B	10	0.125	$m_B r_B$ = 1.25	0.6	0.75	θ_B=0
C	5	0.2	$m_C r_C$ = 1.0	1.2	1.2	θ_C=?
D	4	0.15	$m_D r_D$ = 0.6	1.8	1.08	θ_D=?



(a) Position of planes of masses

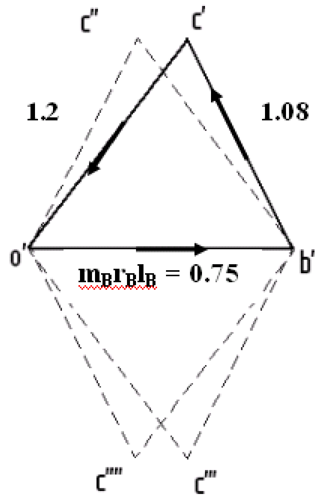


(b) Angular Position of masses

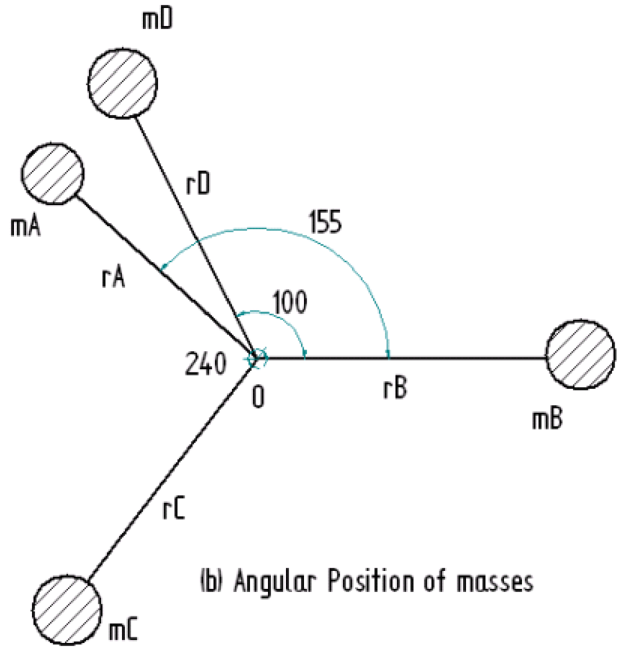
Draw a line o"b" equal to 0.75 kg-m^2 parallel to the line om_B. At point o" and b" draw vectors o"c" and b"c" equal to 1.2 kg-m^2 and 1.08 kg-m^2 respectively. These vectors intersect at point c".

For the construction of force polygon there are four options.

Any one option can be used and relative to that the angular settings of mass C and D are determined.



(c) Couple polygon

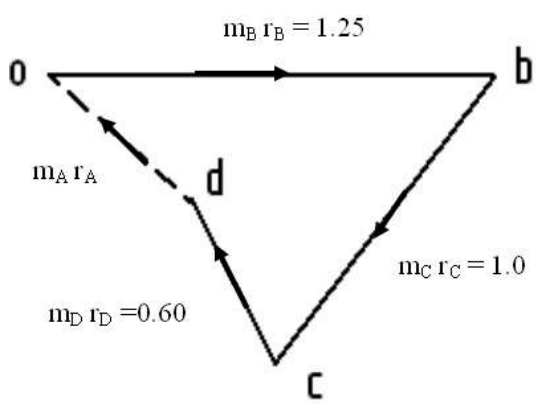


(b) Angular Position of masses

$$\theta_D = 100^0 \quad \text{and} \quad \theta_C = 240^0 \quad \text{Ans}$$

Step 4:

In order to find m_A and its angular setting draw the force polygon as shown in figure (d).



(d) Force polygon

Closing side of the force polygon od represents the product $m_A r_A$. i.e.

$$m_A\ r_A = 0.70\ kg\text{-}m$$

Therefore, $$m_A = \frac{0.70}{r_A} = 7\ kg\ \text{Ans}$$

Step 5:

Now draw line om_A parallel to od of the force polygon. By measurement, we get,

$$\theta_A = 155^0 \quad \text{Ans}$$

Problem 7:

A shaft carries three masses A, B and C. Planes B and C are 60 cm and 120 cm from A. A , B and C are 50 kg, 40 kg and 60 kg respectively at a radius of 2.5 cm. The angular position of mass B and mass C with A are 90^0 and 210^0 respectively. Find the unbalanced force and couple. Also find the position and magnitude of balancing mass required at 10 cm radius in planes L and M midway between A and B, and B andC.

Solution:

Case (i):

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „A" m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	50	0.025	$m_A\ r_A = 1.25$	0	0	$\theta_A = 0^0$
B	40	0.025	$m_B\ r_B = 1.00$	0.6	0.6	$\theta_B = 90^0$
C	60	0.025	$m_C\ r_C = 1.50$	1.2	1.8	$\theta_C = 210^0$

Analytical Method Step

1:

Determination of unbalanced couple

Resolve the couples into their horizontal and vertical components and find their sums. Sum of the horizontal components gives,

$$\sum mrl\ \cos\theta = 0.6\ \cos 90^0 + 1.8 \cos 210^0 = -1.559 \text{ -----(1)}$$

Sum of the vertical components gives,

$$\sum mrl \sin\theta = 0.6 \sin 90^0 + 1.8 \sin 210^0 = -0.3 ------ (2)$$

Squaring and adding (1) and (2), we get

$$C_{unbalanced} = \sqrt{(-1.559)^2 + (-0.3)^2}$$
$$= 1.588 \, kg - m^2$$

Step 2:

Determination of unbalanced force

Resolve the forces into their horizontal and vertical components and find their sums. Sum of the horizontal components gives,

$$\sum mr \cos\theta = 1.25 \cos 0^0 + 1.0 \cos 90^0 + 1.5 \cos 210^0$$
$$= 1.25 + 0 + (-1.299) = -0.049 - - - - - - - - - - (3)$$

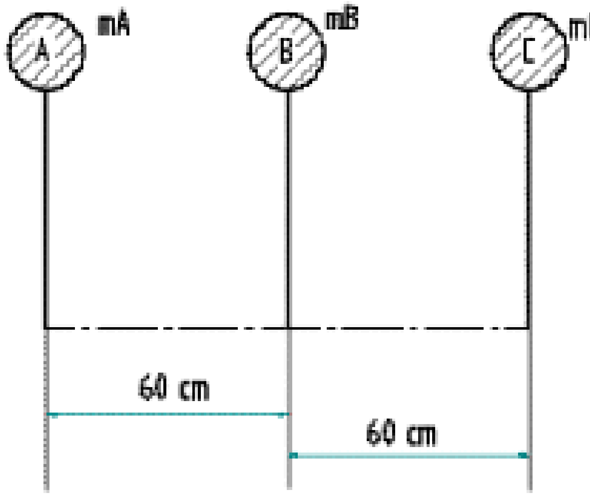
Sum of the vertical components gives,

$$\sum mr \sin\theta = 1.25 \sin 0^0 + 1.0 \sin 90^0 + 1.5 \sin 210^0$$
$$= 0 + 1.0 + (-0.75) = 0.25 - - - - - - - - - - (4)$$

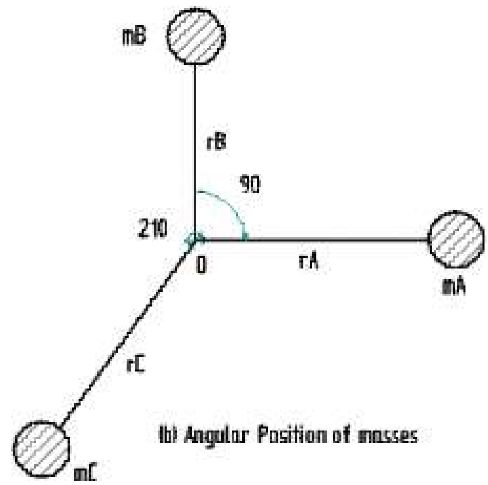
Squaring and adding (3) and (4), we get

$$F_{unbalanced} = \sqrt{(-0.049)^2 + (0.25)^2}$$
$$= 0.2548 \, kg - m$$

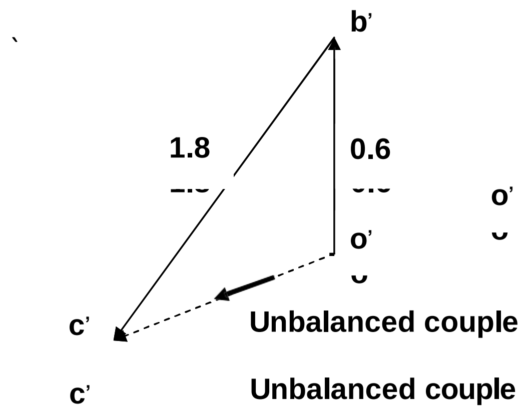
Graphical solution:



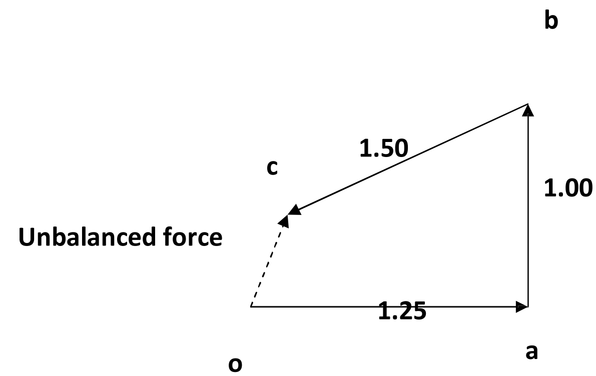
(a) Position of planes of masses



(b) Angular Position of masses

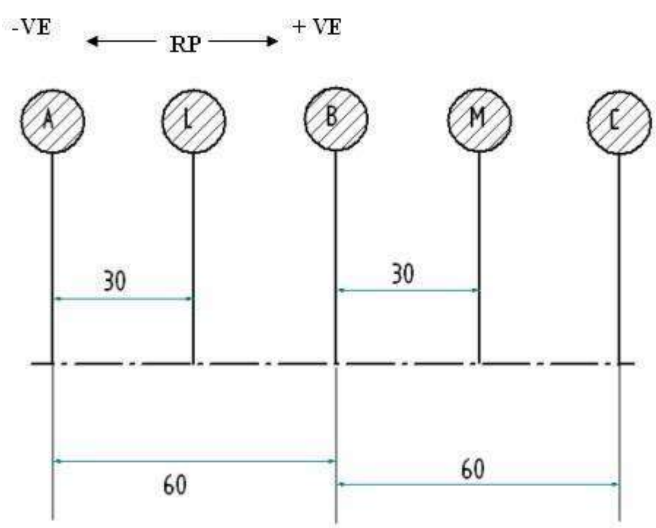


Couple polygon

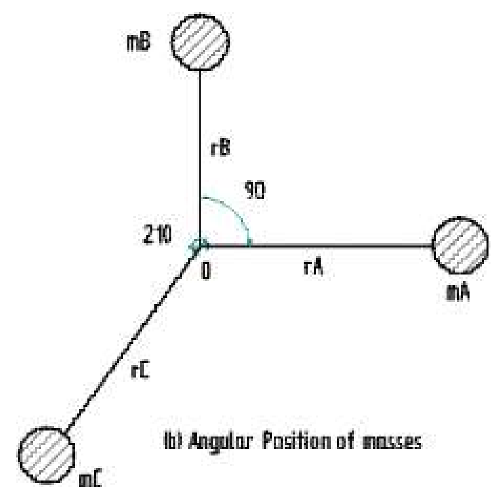


Force polygon

Case (ii):



(a) Position of planes of masses



(b) Angular Position of masses

To determine the magnitude and directions of masses m_M and m_L.

Let, m_L be the balancing mass placed in plane L and m_M be the balancing mass placed in plane M which are to be determined.

The data may be tabulated as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'L' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A	50	0.025	$m_A\ r_A = 1.25$	-0.3	-0.375	$\theta_A = 0^0$
L (R.P.)	m_L = ?	0.10	0.1 m_L	0	0	θ_L =?
B	40	0.025	$m_B\ r_B = 1.00$	0.3	0.3	$\theta_B = 90^0$
M	m_M = ?	0.10	0.1 m_M	0.6	0.06 m_M	θ_M =?
C	60	0.025	$m_C\ r_C = 1.50$	0.9	1.35	$\theta_C = 210^0$

Analytical Method:

Step 1:

Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum m r l \cos\theta + m_M r_M l_M \cos\theta_M = 0$$

On substitution we get

$$-0.375 \cos 0^0 + 0.3 \cos 90^0 + 0.06\ m_M \cos\theta_M + 1.35 \cos 210^0 = 0$$

i.e. $-0.375 + 0 + 0.06\ m_M \cos\theta_M + (-1.16913) = 0$

$$0.06\ m_M \cos\theta_M = 1.54413$$

$$m_M \cos\theta_M = \frac{1.54413}{0.06} = 25.74 \text{ -----(1)}$$

Sum of the vertical components gives,

$$\sum m r l \sin\theta + m_M r_M l_M \sin\theta_M = 0$$

On substitution we get

$$-0.375 \sin 0^0 + 0.3 \sin 90^0 + 0.06\ m_M \sin\theta_M + 1.35 \sin 210^0 = 0$$

i.e. $0 + 0.3 + 0.06\ m_M \sin\theta_M + (-0.675) = 0$

$$0.06\ m_M \sin\theta_M = 0.375$$

$$m_M \sin\theta_M = \frac{0.375}{0.06} = 6.25 \text{ -----(2)}$$

Squaring and adding (1) and (2), we get

$$(m_M \cos \theta_M)^2 + (m_M \sin \theta_M)^2 = (25.74)^2 + (6.25)^2 = 701.61$$

i.e. $m_M^2 = 701.61$ and $m_M = 26.5$ kg Ans

Dividing (2) by (1), we get

$$\tan\theta_M = \frac{6.25}{25.74} \quad \text{and } \theta_M = 13.65^0 \text{ Ans}$$

Step 2:

Resolve the forces into their horizontal and vertical components and find their sums. Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_L r_L \cos\theta_L + m_M r_M \cos\theta_M = 0$$

On substitution w e get

$$1.25 \cos 0^0 + 0.1m_L \cos \theta_L + 1.0 \cos 90^0 + 2.649 \cos 13.65^0 + 1.5 \cos 210^0 = 0$$

$$1.25 + 0.1m_L \cos\theta_L + 0 + 2.5741 + (-1.299) = 0$$

Therefore

$$0.1m_L \cos\theta_L + 2.5251 = 0$$

and $m_L \cos\theta_L = \frac{-2.5251}{0.1} = -25.251$ ----------(3)

Sum of the vertical components gives,

$$\sum mr\sin\theta + m_L r_L \sin\theta_L + m_M r_M \sin\theta_M = 0$$

On substitution we get

$$1.25 \sin 0^0 + 0.1m_L \sin \theta_L + 1.0 \sin 90^0 + 2.649 \sin 13.65^0 + 1.5 \sin 210^0 = 0$$

$$0 + 0.1m_L \sin\theta_L + 1 + 0.6251 + (-0.75) = 0$$

Therefore

$$0.1m_L \sin\theta_L + 0.8751 = 0$$

and $m_L \sin\theta_L = \frac{-0.8751}{0.1} = -8.751$ ----------(4)

Squaring and adding (3) and (4), we get

$$(m_L \cos\theta_L)^2 + (m_L \sin\theta_L)^2 = (-25.251)^2 + (-8.751)^2 = 714.193$$

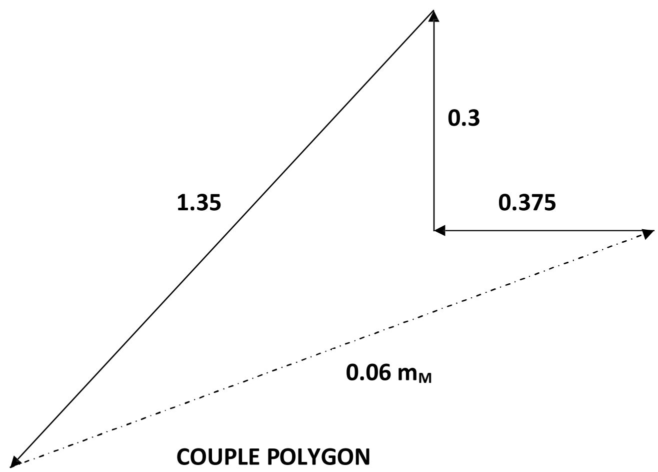
$$\text{i.e. } m_L^2 = 714.193 \qquad \text{and} \qquad m_L = 26.72 \text{ kg} \quad \text{Ans}$$

Dividing (4) by (3), we get

$$\tan\theta_L = \frac{-8.751}{-25.251} \quad \text{and} \quad \theta_L = 19.11^0$$

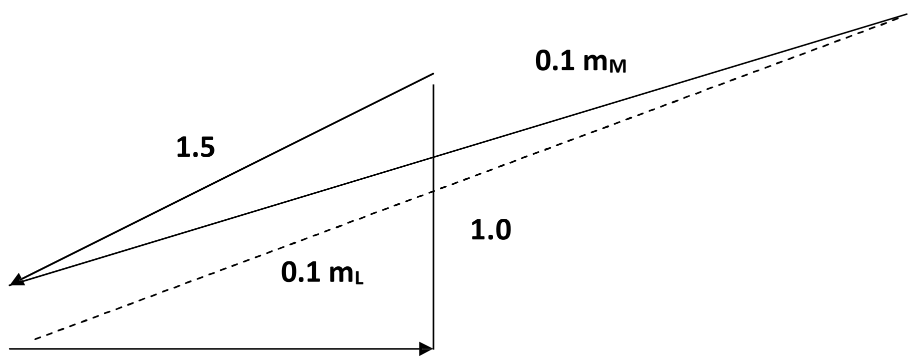
Ans

The balancing mass m_L is at an angle $19.11^0 + 180^0 = 199.11^0$ measured in counter clockwise direction.



COUPLE POLYGON

Graphical Method:



FORCE POLYGON

Problem 8:

Four masses A, B, C and D are completely balanced. Masses C and D make angles of 90^0 and 210^0 respectively with B in the same sense. The planes containing B and C are 300 mm apart. Masses A, B, C and D can be assumed to be concentrated at radii of 360 mm, 480 mm, 240 mm and 300 mm respectively. The masses B, C and D are 15 kg, 25 kg and 20 kg respectively. Determine i) mass A and its angular position ii) position of planes A and D.

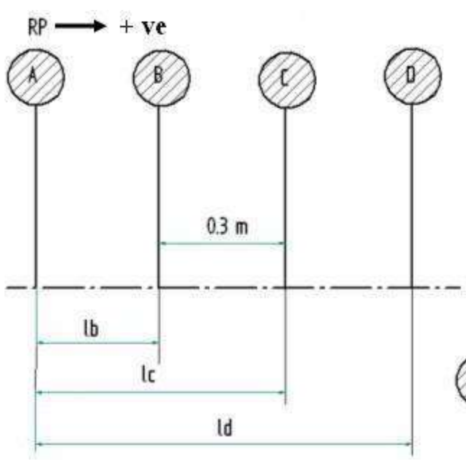
Solution: Analytical Method

Step 1:

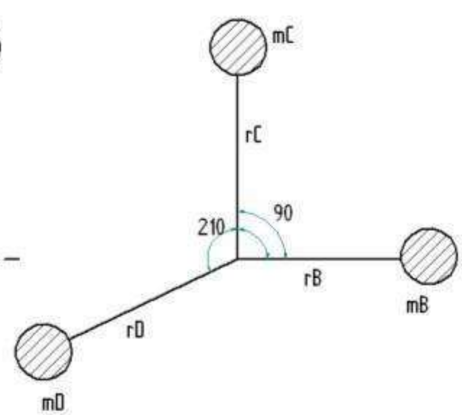
Draw the space diagram or angular position of the masses. Since the angular position of the masses C and D are given with respect to mass B, take the angular position of mass B as $\theta_B = 0^0$.

Tabulate the given data as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane „A" m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	m_A= ?	0.36	$m_A r_A$ = 0.36 m_A	0	0	θ_A=**?**
B	15	0.48	$m_B r_B$ = 7.2	l_B = ?	7.2 l_B	θ_B=0
C	25	0.24	$m_C r_C$ = 6.0	l_C = ?	6.0 l_C	θ_C =90^0
D	20	0.30	$m_D r_D$ = 6.0	l_D = ?	6.0 l_D	θ_D = 210^0



(a) Position of planes of masses (Assumed)



(b) Angular position of masses

Step 2:

Mass m_A be the balancing mass placed in plane A which is to be determined along with its angular position.

Refer column 4 of the table. Since m_A is to be determined (which is the only unknown) ,resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D = 0$$

On substitution we get

$$0.36\, m_A \cos\theta_A + 7.2\cos 0^0 + 6.0\cos 90^0 + 6.0\cos 210^0 = 0$$

Therefore

$$0.36\, m_A \cos\theta_A = -2.004 \text{----------} (1)$$

Sum of the vertical components gives,

$$\sum mr\sin\theta = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D = 0$$

On substitution we get

$$0.36\, m_A \sin\theta_A + 7.2\sin 0^0 + 6.0\sin 90^0 + 6.0\sin 210^0 = 0$$

Therefore

$$0.36\, m_A \sin\theta_A = -3.0 \text{----------} (2)$$

Squaring and adding (1) and (2), we get

$$0.36^2 (m_A)^2 = (-2.004)^2 + (-3.0)^2 = 13.016$$

$$m_A = \sqrt{\frac{13.016}{0.36^2}} = 10.02 \text{ kg Ans}$$

Dividing (2) by (1), we get

$$\tan\theta_A = \frac{-3.0}{-2.004}$$ and Resutltant makes an angle $= 56.26^0$

The balancing mass A makes an angle of $\theta_A = 236.26^0$ Ans

Step 3:

Resolve the couples into their horizontal and vertical components and find their sums. Sum of the horizontal components gives,

$$\sum mrl\cos\theta = m_A r_A l_A \cos\theta_A + m_B r_B l_B \cos\theta_B + m_C r_C l_C \cos\theta_C + m_D r_D l_D \cos\theta_D = 0$$

On substitution we get

$$0 + 7.2l_B \cos 0^0 + 6.0l_C \cos 90^0 + 6.0l_D \cos 210^0 = 0$$

$$7.2l_B - 5.1962l_D = 0 \text{ ------------------(3)}$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta = m_A r_A l_A \sin\theta_A + m_B r_B l_B \sin\theta_B + m_C r_C l_C \sin\theta_C + m_D r_D l_D \sin\theta_D = 0$$ **On substitution we get**

$$0 + 7.2l_B \sin 0^0 + 6.0l_C \sin 90^0 + 6.0l_D \sin 210^0 = 0$$

$$0 + 0 + 6.0l_C - 3l_D = 0 \text{ ------------------ (4)}$$

But from figure we have, $l_C = l_B + 0.3$

On substituting this in equation (4), we get

$$6.0(l_B + 0.3) - 3l_D = 0$$

$$\text{i.e. } 6.0l_B - 3l_D = 1.8 \text{ ----------------- (5)}$$

Thus we have two equations (3) and (5), and two unknowns l_B, l_D

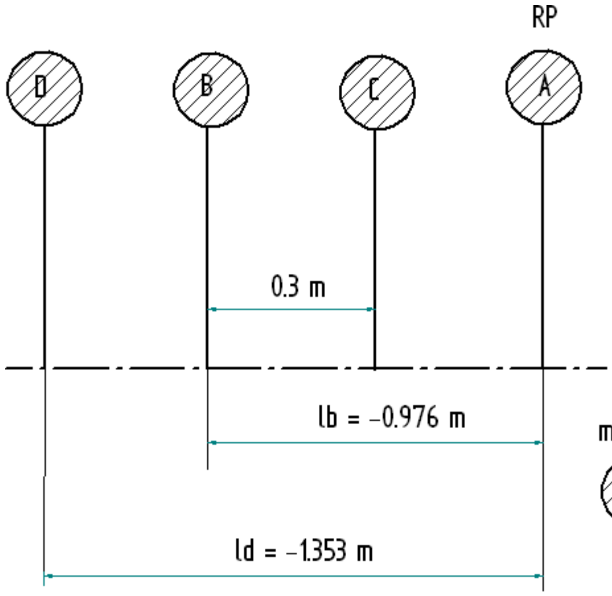
$$7.2l_B - 5.1962l_D = 0 \text{ -------------------(3)}$$

$$6.0l_B - 3l_D = 1.8 \text{ - - - - - - - - - - -(5)}$$

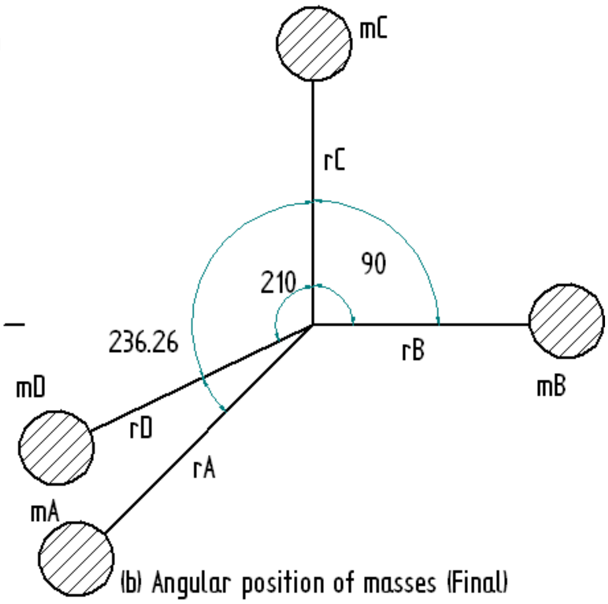
On solving the equations, we get

$l_D = -1.353\,m$ and $l_B = -0.976\,m$

As per the position of planes of masses assumed the distances shown are positive (+ ve) from the reference plane A. But the calculated values of distances l_B and l_D are negative. The corrected positions of planes of masses is shown below.



(a) Position of planes of masses (Corrected)



(b) Angular position of masses (Final)

BALANCING OF RECRIPROCATING MASSES

SLIDER CRANK MECHANISM:

PRIMARY AND SEC ONDARY ACCELERATING FORCE:

Acceleration of the reciprocating mass of a slider-crank mechanism is given by,

$$a_p = \text{Acceleration of piston}$$

$$= r\omega^2 \left[\cos\theta + \frac{\cos 2\theta}{n} \right] ---------- (1)$$

$$\text{Where } n = \frac{l}{r}$$

And, the force required to accelerate the mass ‘m’ is

$$F_i = mr\omega^2 \left[\cos\theta + \frac{\cos 2\theta}{n} \right]$$

$$= mr\omega^2 \cos\theta + mr\omega^2 \frac{\cos 2\theta}{n} ---------- (2)$$

The first term of the equation (2) , i.e. $mr\omega^2\cos\theta$ is called **primary accelerating force** the second term $mr\omega^2\frac{\cos 2\theta}{n}$ is called the **secondary accelerating force.**

Maximum value of primary accelerating force is $mr\omega^2$

And Maximum value of secondary accelerating force is $\frac{mr\omega^2}{n}$

Generally, ‘n’ value is much greater than one; the secondary force is small compared to primary force and can be safely neglected for slow speed engines.

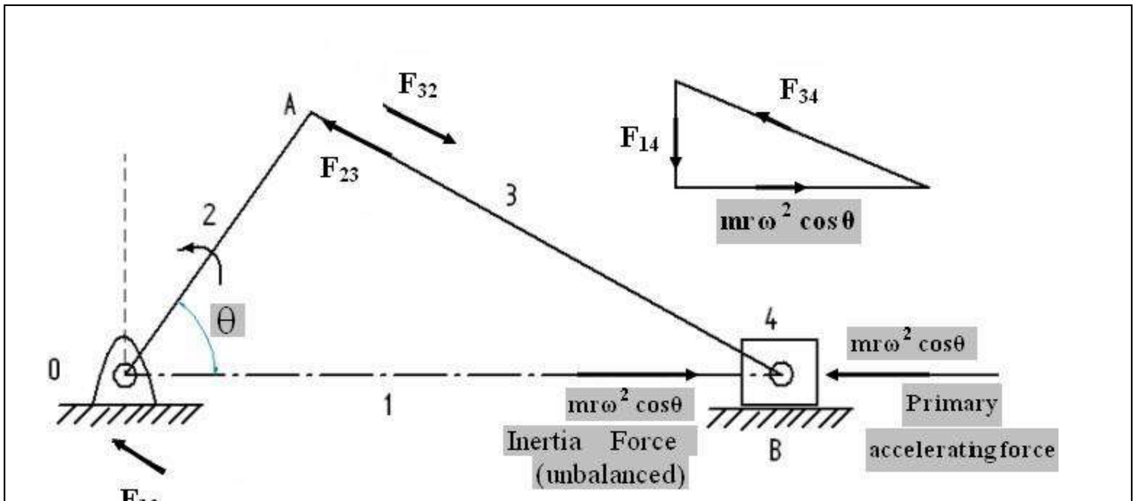


Figure (a)

In Fig (a), the inertia force due to primary accelerating force is shown.

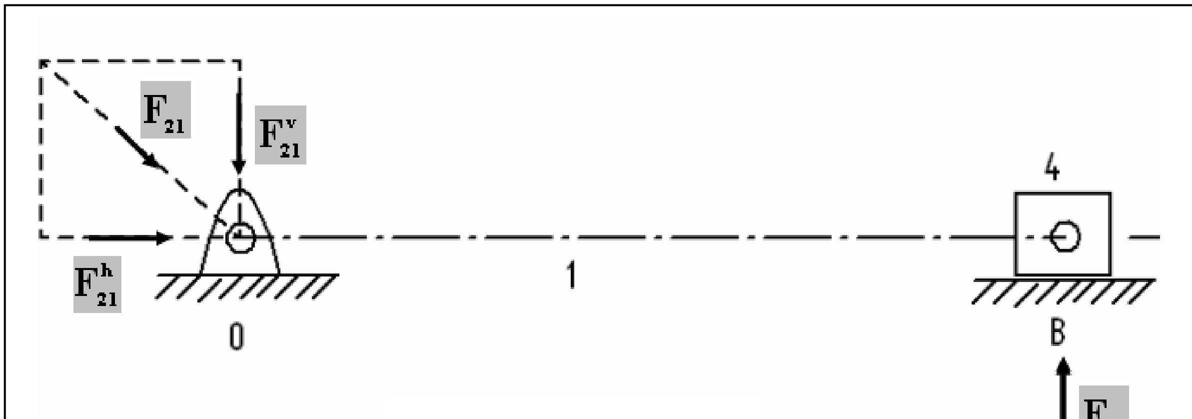


Figure (b)

In Fig (b), the forces acting on the engine frame due to inertia force are shown.

At 'O' the force exerted by the crankshaft on the main bearings has two components,

horizontal F_2^h and vertical F_{21}^v

F_{21}^h is an horizontal force, which is an **unbalanced shaking force.**

F_{21}^v and F_{41}^v balance each other but form an **unbalanced shaking couple**.

The magnitude and direction of these unbalanced force and couple go on changing with angle θ. The shaking force produces linear vibrations of the frame in horizontal direction, whereas the shaking couple produces an oscillating vibration.

The shaking force F_{21}^h is the only unbalanced force which may hamper the smooth running of the engine and effort is made to balance the same.

However it is not at all possible to balance it completely and only some modifications can be carried out.

BALANCING OF THE SHAKING FORCE:

Shaking force is being balanced by adding a rotating counter mass at radius 'r' directly opposite the crank. This provides only a partial balance. This counter mass is in addition to the mass used to balance the rotating

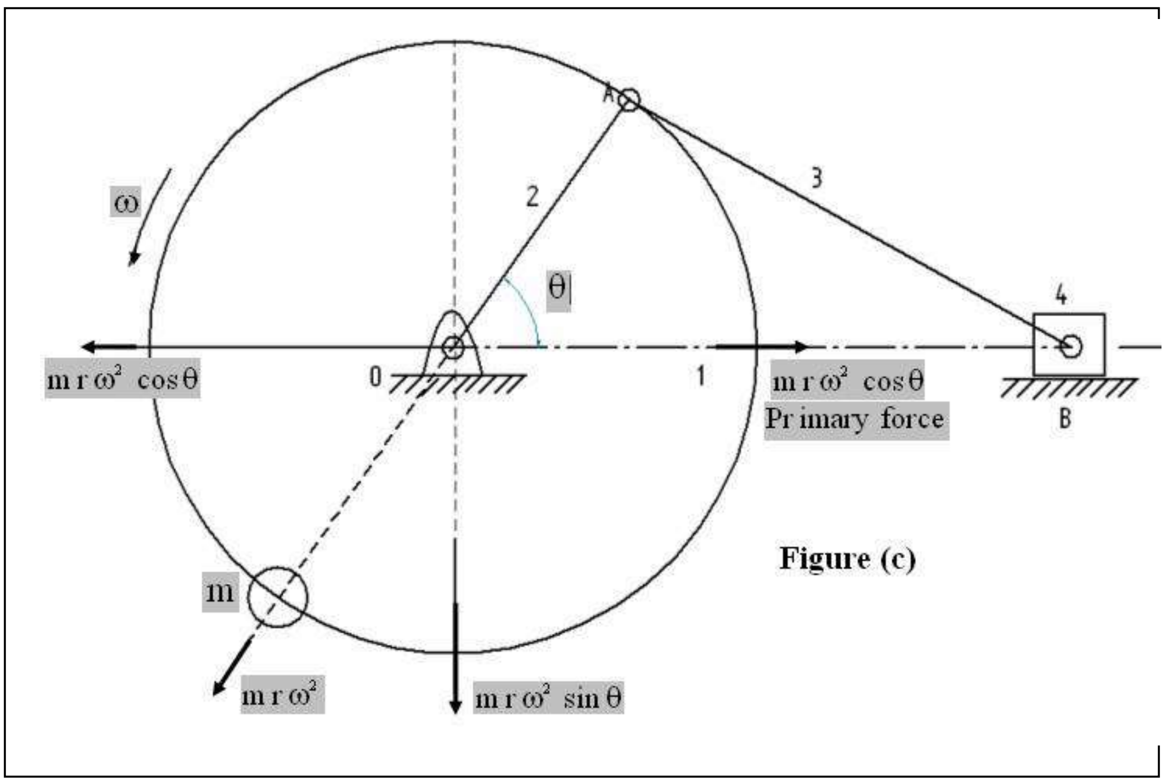


Figure (c)

unbalance due to the mass at the crank pin. This is shown in figure (c).

The horizontal component of the centrifugal force due to the balancing mass is $mr\ \omega^2 \cos\theta$ and this is in the line of stroke. This component neutralizes the unbalanced reciprocating force. But the rotating mass also has a component $mr\omega^2 \sin\theta$ perpendicular to the line of stroke which remains unbalanced. The unbalanced force is zero at $\theta = 0^0$ or 180^0 and maximum at the middle of the stroke i.e. $\theta = 90^0$. The magnitude or the maximum value of the unbalanced force remains the same i.e. equal to $mr\omega^2$. Thus instead of sliding to and fro on its mounting, the mechanism tends to jump up and down.

To minimize the effect of the unbalance force a compromise is, usually made, is $\frac{2}{3}$ of the reciprocating mass is balanced or a value between $\frac{1}{2}$ to $\frac{3}{4}$.

If 'c' is the fraction of the reciprocating mass, then

$$\textbf{The primary force balanced by the mass} = cmr\omega^2 \cos\theta$$

and

$$\textbf{The primary force unbalanced by the mass} = (1-c)\, mr\omega^2 \cos\theta$$

$$\textbf{Vertical component of centrifugal force which remains unbalanced}$$

$$= c\ mr\ \omega^2 \sin\theta$$

In reciprocating engines, unbalance forces in the direction of the line of stroke are more dangerous than the forces perpendicular to the line of stroke.

$$\textbf{Resultant unbalanced force at any instant}$$

$$= \sqrt{\left[(1-c)mr\omega^2 \cos\theta\right]^2 + \left[cmr\omega^2 \sin\theta\right]^2}$$

The resultant unbalanced force is minimum when, $c = \frac{1}{2}$

This method is just equivalent to as if a revolving mass at the crankpin is completely balanced by providing a counter mass at the same radius diametrically opposite to the crank. Thus if m_P is the mass at the crankpin and ‘c’ is the fraction of the reciprocating mass ‘m’ to be balanced , the mass at the crankpin may be considered as $cm + m_P$ which is to be completely balanced.

Problem 1:

A single –cylinder reciprocating engine has a reciprocating mass of 60 kg. The crank rotates at 60 rpm and the stroke is 320 mm. The mass of the revolving parts at 160 mm radius is 40 kg. If two-thirds of the reciprocating parts and the whole of the revolving parts are to be balanced, determine the, (i) balance mass required at a radius of 350 mm and (ii) unbalanced force when the crank has turned 50^0 from the top-dead centre.

Solution:

Given: $m = \text{mass of the reciprocating parts} = 60\ \text{kg}$

$N = 60\ \text{rpm},\ L = \text{length of the stroke} = 320\ \text{mm}$

$m_p = 40\ \text{kg},\ c = \frac{2}{3},\ r_c = 350\ \text{mm}$

(i) Balance mass required at a radius of 350 mm

We have,

$$\omega = \frac{2\pi N}{60} = \frac{2\pi x60}{60} = 2\pi\ \text{rad/s}$$

$$r = \frac{L}{2} = \frac{320}{2} = 160\text{mm}$$

Mass to be balanced at the crank pin = M

$$M = c\,m + m_p = \frac{2}{3} x60 + 40 = 80\ \text{kg}$$

and

$$m_c r_c = Mr \quad \text{therefore} \quad m_c = \frac{Mr}{r_c}$$

$$\text{i.e. } m_c = \frac{80x160}{350} = 36.57\ \text{kg}$$

(ii) Unbalanced force when the crank has turned 50^0 from the top-dead centre.

Unbalanced force at $\theta = 50^0$

$$= \sqrt{\left[(1-c)mr\omega^2\cos\theta\right]^2 + \left[cmr\omega^2\sin\theta\right]^2}$$

$$\sqrt{\left[\left(1-\frac{2}{3}\right)x60x0.16x(2\pi)^2\cos 50^0\right]^2 + \left[\frac{2}{3}x60x0.16x(2\pi)^2\sin 50^0\right]^2}$$

$=$

$= \mathbf{209.9N}$

Problem 2:

The following data relate to a single cylinder reciprocating engine: Mass of reciprocating parts = 40 kg

Mass of revolving parts = 30 kg at crank radius Speed = 150 rpm, Stroke = 350 mm.

If 60 % of the reciprocating parts and all the revolving parts are to be balanced, determine the,

(i) balance mass required at a radius of 320 mm and (ii) unbalanced force when the crank has turned 45^0 from the top-dead centre.

Solution:

Given : $\mathbf{m = mass\ of\ the\ reciprocating\ parts = 40\ kg}$

$$\mathbf{m_P = 30\ kg\ ,\ N = 150\ rpm,\ L = length\ of\ the\ stroke = 350\ mm}$$

$$\mathbf{c = 60\%\ ,\ r_c = 320\ mm}$$

(i) Balance mass required at a radius of 350 mm

We have,

$$\boldsymbol{\omega} = \frac{\mathbf{2\pi N}}{\mathbf{60}} = \frac{\mathbf{2\pi x150}}{\mathbf{60}} = \mathbf{15.7\ rad/s}$$

$$\mathbf{r = \frac{L}{2} = \frac{350}{2} = 175\ mm}$$

$$\mathbf{Mass\ to\ be\ balanced\ at\ the\ crank\ pin = M}$$

$$\mathbf{M = c\,m + m_P = 0.60\ x40 + 30 = 54\ kg}$$

and

$$\mathbf{m_c\, r_c = Mr \quad therefore \quad m_c = \frac{Mr}{r_c}}$$

$$\mathbf{i.e.\ m_c = \frac{54x175}{320} = 29.53\ kg}$$

(ii) Unbalanced force when the crank has turned 45^0 from the top-dead centre.

Unbalanced force at $\boldsymbol{\theta = 45^0}$

$$= \sqrt{\left[(1-c)mr\,\omega^2 \cos\theta\right]^2 + \left[c\,mr\,\omega^2 \sin\theta\right]^2}$$

$$= \sqrt{\left[(1-0.60)\,x40\,x0.175\,x(15.7)^2 \cos 45^0\right]^2 + \left[0.60\,x40\,x0.175\,x(15.7)^2 \sin 45^0\right]^2}$$

$= \mathbf{880.7\ N}$

SECONDARY BALANCING:

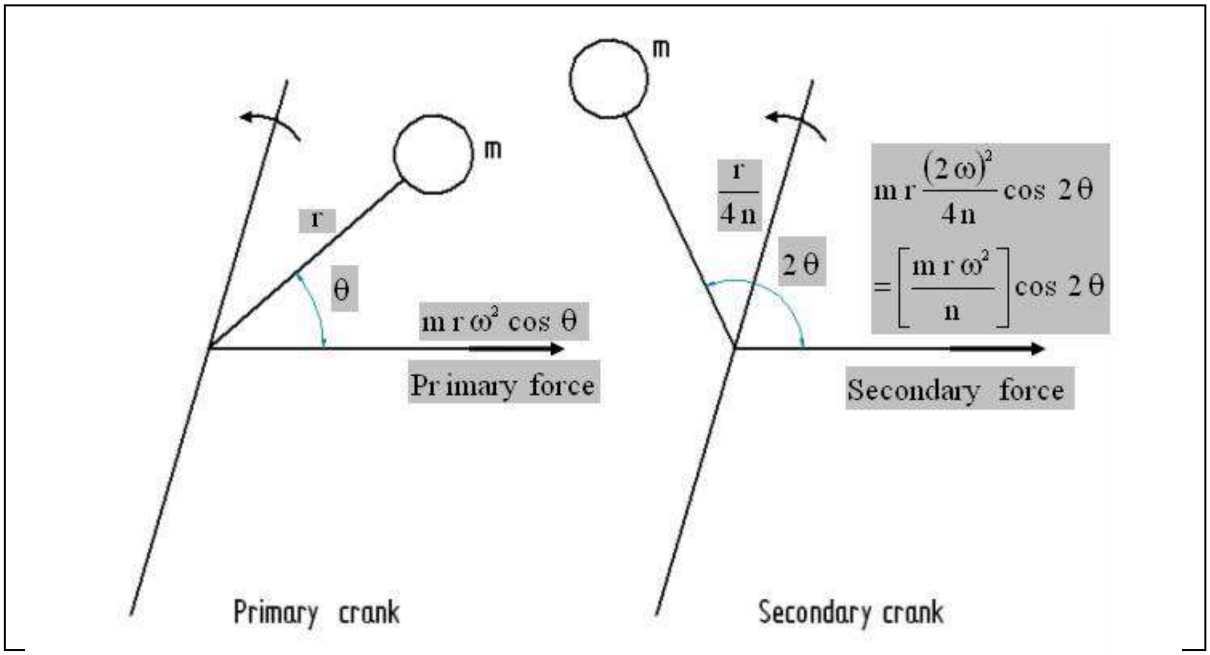
Secondary acceleration force is equal to $\mathbf{mr}\,\omega^2 \dfrac{\cos 2\theta}{\mathbf{n}}$ – – – – – – –(1)

Its frequency is twice that of the primary force and the magnitude magnitude of the primary $1/\mathbf{n}$ times the force.

The secondary force is also equal to $\mathbf{mr}(2\omega)^2 \dfrac{\cos 2\theta}{4\mathbf{n}}$ – – – – – – –(2)

Consider, two cranks of an engine, one actual one and the other imaginary with the following specifications.

	Actual	Imaginary
Angular velocity	ω	2ω
Length of crank	$\mathbf{r}$	$\dfrac{\mathbf{r}}{4\mathbf{n}}$
Mass at the crank pin	m	m



Thus, when the actual crank has turned through an angle would have $\theta = \omega\, t$, the imaginary crank turned an angle $2\theta = 2\omega\, \mathbf{t}$

Centrifugal force induced in the imaginary crank = $\frac{mr(2\omega)^2}{4n}$

Component of this force along the line of stroke is = $\frac{mr(2\omega)^2}{4n}\cos 2\theta$

Thus the effect of the secondary force is equivalent to an imaginary crank of length $\frac{r}{4n}$ rotating at double the angular velocity, i.e. twice of the engine speed. The imaginary crank coincides with the actual at inner top-dead centre. At other times, it makes an angle with the line of stroke equal to twice that of the engine crank.

The secondary couple about a reference plane is given by the multiplication of the secondary force with the distance ' l ' of the plane from the reference plane.

COMPLETE BALANCING OF RECIPROCATING PARTS

Conditions to be fulfilled:

1. Primary forces must balance i.e., primary force polygon is enclosed.
2. Primary couples must balance i.e., primary couple polygon is enclosed.
3. Secondary forces must balance i.e., secondary force polygon is enclosed.
4. Secondary couples must balance i.e., secondary couple polygon is enclosed.

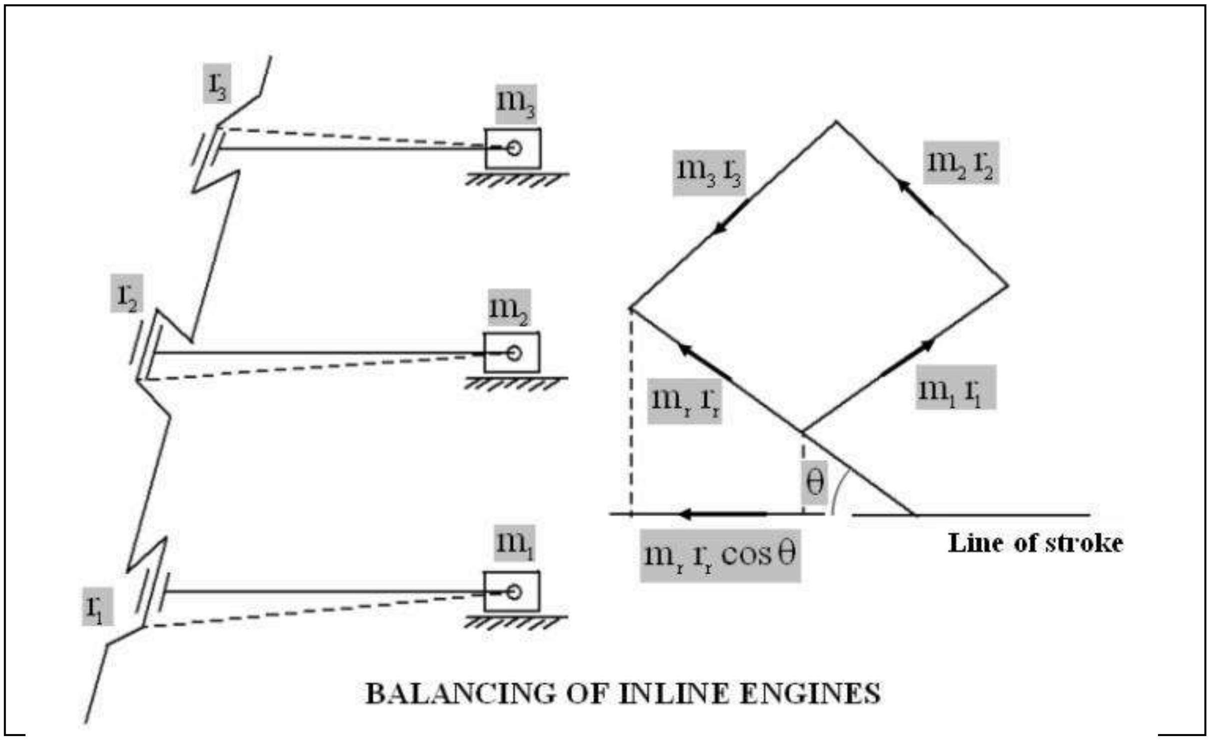
Usually, it is not possible to satisfy all the above conditions fully for multi-cylinder engine. Mostly some unbalanced force or couple would exist in the reciprocating engines.

BALANCING OF INLINE ENGINES:

An in-line engine is one wherein all the cylinders are arranged in a single line, one behind the other. Many of the passenger cars such as Maruti 800, Zen, Santro, Honda-city, Honda CR-V, Toyota corolla are the examples having four cinder in-line engines.

In a reciprocating engine, the reciprocating mass is transferred to the crankpin; the axial component of the resulting centrifugal force parallel to the axis of the cylinder is the primary unbalanced force.

Consider a shaft consisting of three equal cranks asymmetrically spaced. The crankpins carry equivalent of three unequal reciprocating masses, then



BALANCING OF INLINE ENGINES

Primary **force** $= \sum mr\omega^2 \cos\theta$ ———————————— **(1)**

Primary couple $= \sum mr\omega^2\, l\cos\theta$ ———————————— **(2)**

Secondary force $= \sum m\, r \frac{(2\omega)^2}{4n} \cos 2\theta$ ———————————— **(3)**

And Secondary couple $= \sum m\, r \frac{(2\omega)^2}{4n}\, l \cos 2\theta$

$= \sum m\, r \frac{\omega^2}{n}\, l \cos 2\theta$ ———————————— **(4)**

GRAPHICAL SOLUTION:

To solve the above equations graphically, first draw the $\sum m\, r \cos\theta$ polygon (ω^2 is common to all forces). Then the axial component of the resultant forces $(F_r \cos\theta)$ multiplied by ω^2 provides the primary unbalanced force on the system at that moment. This unbalanced force is zero when $\theta = 90^0$ and a maximum when $\theta = 0^0$.

If the force polygon encloses, the resultant as well as the axial component will always be zero and the system will be in **primary balance**.
Then,

$$\sum F_{Ph} = 0 \text{ and } \sum F_{PV} = 0$$

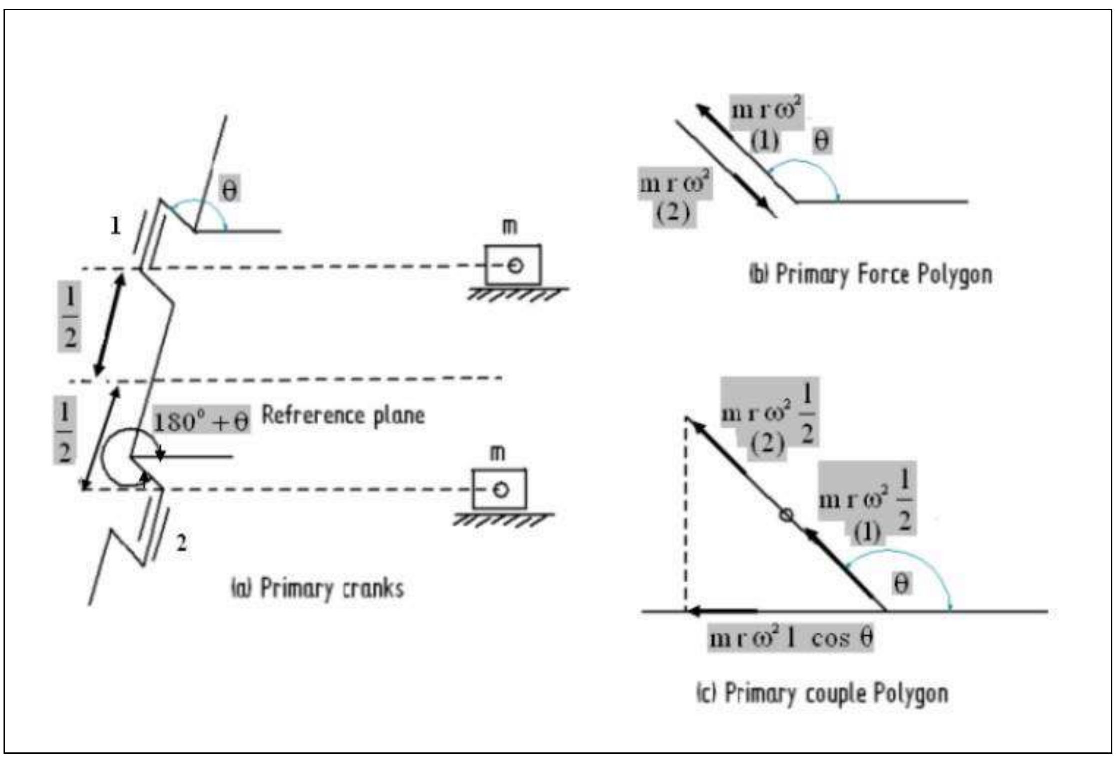
To find the secondary unbalance force, first find the positions of the imaginary secondary cranks. Then transfer the reciprocating masses and multiply the same by $\frac{(2\omega)^2}{4n}$ or $\frac{\omega^2}{n}$ to get the secondary force.

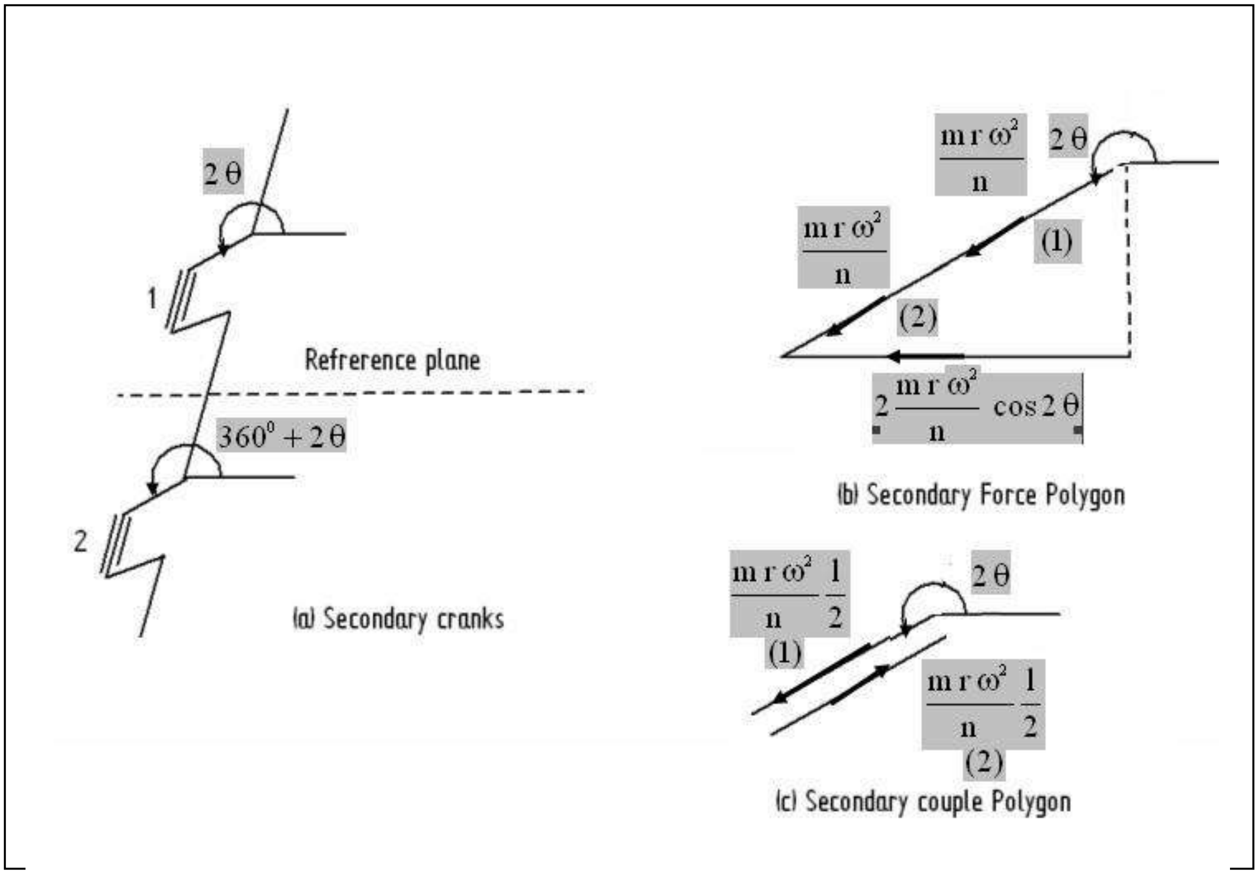
In the same way primary and secondary couple (m r l) polygon can be drawn for primary and secondary couples.

Case 1:

IN-LINE TWO-CYLINDER ENGINE

Two-cylinder engine, cranks are 180^0 apart and have equal reciprocating masses.





(a) Secondary cranks

(b) Secondary Force Polygon

(c) Secondary couple Polygon

Taking a plane through the **centre line** as the reference plane,

Primary **force** $= m r\omega^2 \left[\cos\theta + \cos(180 + \theta)\right] = 0$

Primary couple $= m r\omega^2 \left[\frac{l}{2}\cos\theta + \left(-\frac{l}{2}\right)\cos(180 + \theta)\right] = m r\omega^2 l \cos\theta$

Maximum values are $m r\omega^2 l$ at $\theta = 0^0$ and 180^0

Secondary **force** $= \frac{m r\omega^2}{n}\left[\cos 2\theta + \cos(360 + 2\theta)\right] = \frac{2m r\omega^2}{n}\cos 2\theta$

Maximum values are $\frac{2m r\omega^2}{n}$ when $2\theta = 0^\circ, 180^\circ, 360^\circ$ and 540°

or $\theta = 0^\circ, 90^\circ, 180^\circ$ and 270°

$$\textbf{Secondary couple} = \frac{m r \omega^2}{n}\left[\frac{l}{2}\cos 2\theta + \left(-\frac{l}{2}\right)\cos(360 + 2\theta)\right] = 0$$

ANALYTICAL METHOD OF FINDING PRIMARY FORCES AND COUPLES

- First the positions of the cranks have to be taken in terms of θ^{o}
- The maximum values of these forces and couples vary instant to instant and are equal to the values as given by the equivalent rotating masses at the crank pin.

If a particular position of the crank shaft is considered, the above expressions may not give the maximum values.
For example, the maximum value of primary couple is $m r \omega^2 l$ and this value is obtained at crank positions 0^0 and 180^0. However, if the crank positions are assumed at 90^0 and 270^0, the values obtained will be zero.

- If any particular position of the crank shaft is considered, then both X and Y components of the force and couple can be taken to find the maximum values.

For example, if the crank positions considered as 120^0 and 300^0, the primary couple can be obtained as

$$\textbf{X-component} = m r \omega^2 \left[\frac{l}{2}\cos 120^o + \left(-\frac{l}{2}\right)\cos\left(180^o + 120^o\right)\right]$$

$$= -\frac{1}{2} m r \omega^2 l$$

$$\textbf{Y-component} = m r \omega^2 \left[\frac{l}{2}\sin 120^o + \left(-\frac{l}{2}\right)\sin\left(180^o + 120^o\right)\right]$$

$$= \frac{\sqrt{3}}{2} m r \omega^2 l$$

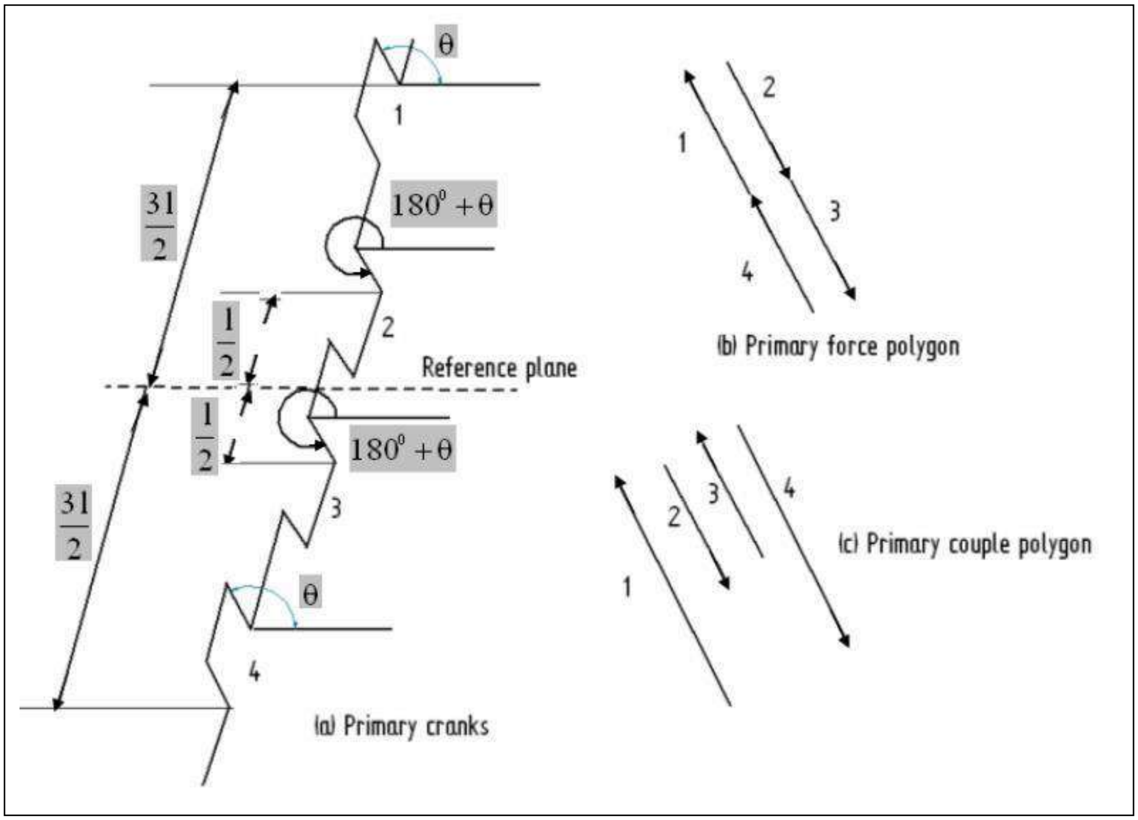
Therefore, $$\textbf{Primary couple} = \sqrt{\left(-\frac{1}{2} m r \omega^2 l\right)^2 + \left(\frac{\sqrt{3}}{2} m r \omega^2 l\right)^2}$$

$$= m r \omega^2 l$$

Case 2:

IN-LINE FOUR-CYLINDER FOUR-STROKE ENGINE

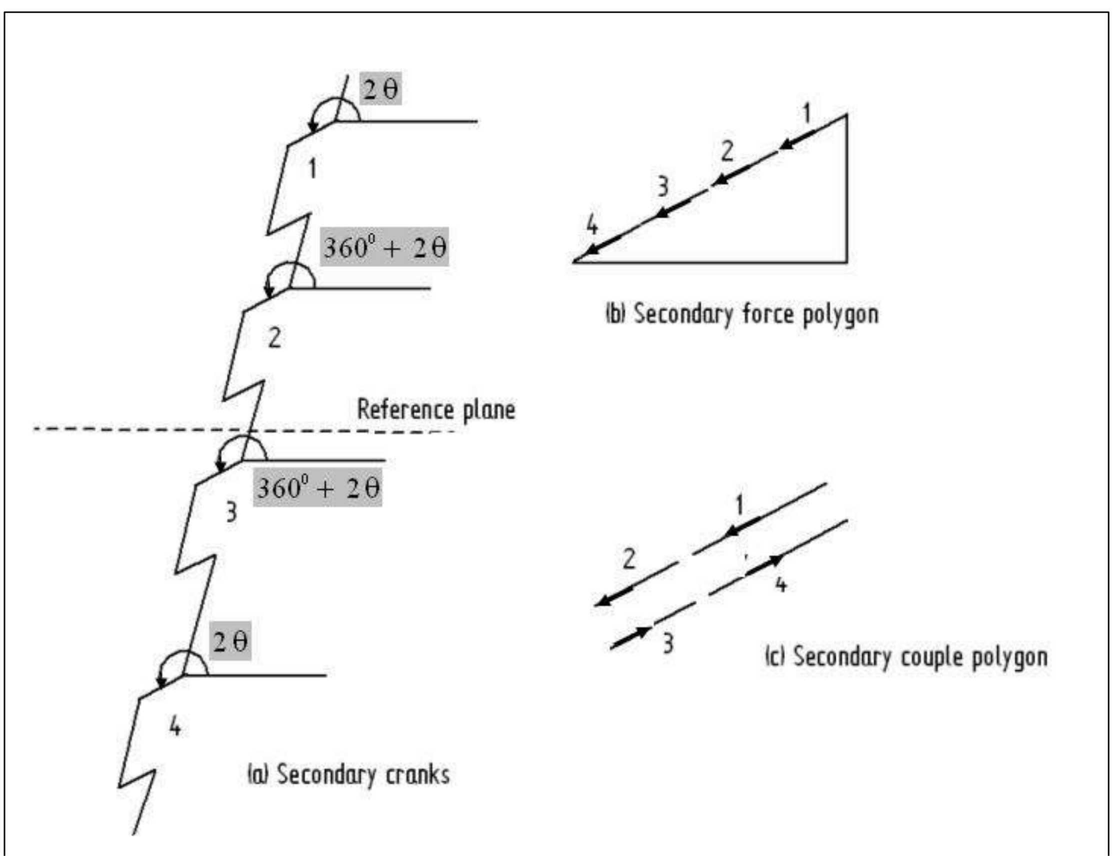
This engine has tow outer as well as inner cranks (throws) in line. The inner throws are at 180^0 to the outer throws. Thus the angular positions for the cranks are θ^0 for the first, $180^0 + \theta^0$ for the second, $180^0 + \theta^0$ for the third and θ^0 for the fourth.



(a) Primary cranks

(b) Primary force polygon

(c) Primary couple polygon



(a) Secondary cranks

(b) Secondary force polygon

(c) Secondary couple polygon

FINDING PRIMARY FORCES, PRIMARY COUPLES, SECONDARY FORCES AND SECONDARY COUPLES:

Choose a plane passing through the middle bearing about which the arrangement is symmetrical as the reference plane.

$$\textbf{Primary force} = m r\omega^2 \left[\cos\theta + \cos(180^0+\theta) + \cos(180^0+\theta) + \cos\theta\right]$$

$$= 0$$

$$\textbf{Primary couple} = m r\omega^2 \left[\frac{3l}{2}\cos\theta + \frac{l}{2}\cos(180^0+\theta) + \left(-\frac{l}{2}\right)\cos(180^0+\theta) + \left(-\frac{3l}{2}\right)\cos\theta\right]$$

$$= 0$$

$$\textit{Secondary}\ \textbf{force} = \frac{m r \omega^2}{n}\left[\cos 2\theta + \cos(360^0+2\theta) + \cos(360^0+2\theta) + \cos 2\theta\right]$$

$$= \frac{4m r\omega^2}{n}\cos 2\theta$$

$$\textbf{Maximum value} = \frac{m r\omega^2}{n}$$

$$\text{at } 2\theta = 0^0, 180^0, 360^0 \text{ and } 540^0 \text{ or}$$

$$\theta = 0^0, 90^0, 180^0 \text{ and } 270^0$$

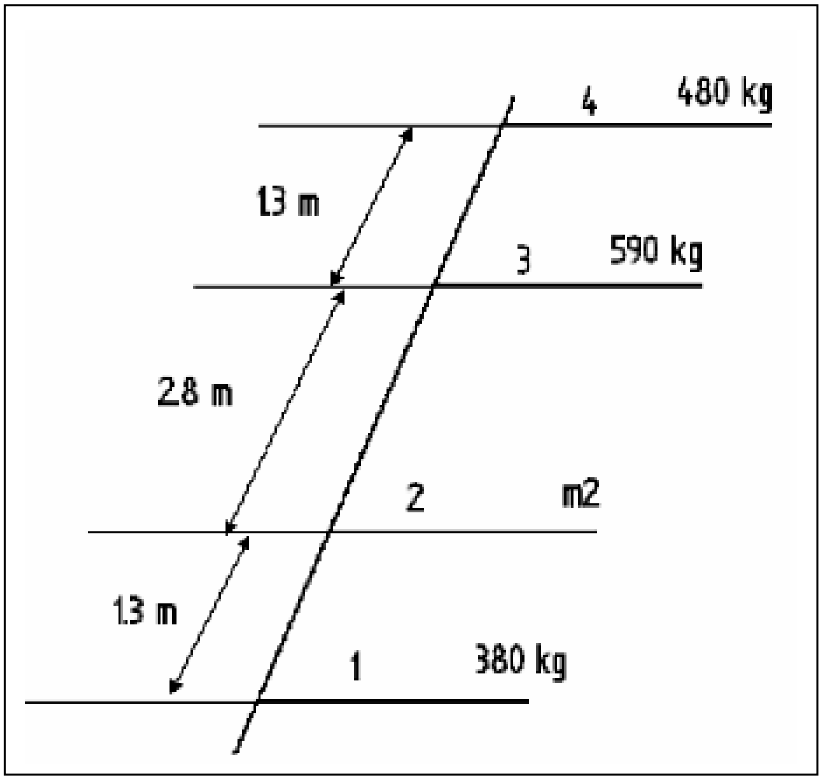
$$m r\omega^2 \left[\frac{3l}{2}\cos 2\theta + \frac{l}{2}\cos(360^0+2\theta)\right.$$

Secondary **couple** $= \frac{n}{\quad} \left[\ldots + \left(-\frac{l}{2} \right) \cos(360^0 + 2\theta) + \left(-\frac{3l}{2} \right) \cos 2\theta \right] = 0$

Thus the engine is not balanced in secondary forces.

Problem 1:

A four-cylinder oil engine is in complete primary balance. The arrangement of the reciprocating masses in different planes is as shown in figure. The stroke of each piston is 2 r mm. Determine the reciprocating mass of the cylinder 2 and the relative crank position.



Solution:

Given :

$m_1 = 380 \text{ kg}, \; m_2 = ?, \; m_3 = 590 \text{ kg}, \; m_4 = 480 \text{ kg}$

$\text{crank length} = \frac{L}{2} = \frac{2r}{2} = r$

Plane	Mass (m) kg	Radius (r) m	Cent. Force/ω^2 (m r) kg m	Distance from Ref plane '2' m	Couple/ ω^2 (m r l) kg m^2
1	380	r	380 r	-1.3	-494 r
2(RP)	m_2	r	m_2 r	0	0
3	590	r	590 r	2.8	1652 r
4	480	r	480 r	4.1	1968 r

Analytical Method:

Choose plane 2 as the reference plane and $\theta_3 = 0^0$.

.

Step 1:

Resolve the couples into their horizontal and vertical components and take their sums. Sum of the horizontal components gives

$$-494\, r \cos\theta_1 + 1652\, r\cos 0^0 + 1968\, r\cos\theta_4 = 0$$

i.e., $$+494 \cos\theta_1 = 1652 + 1968 \cos\theta_4 ---------- (1)$$

Sum of the vertical components gives

$$-494\, r \sin\theta_1 + 1652\, r\sin 0^0 + 1968\, r\sin\theta_4 = 0$$

i.e., $$494 \sin\theta_1 = 1968 \sin\theta_4 ---------- (2)$$

Squaring and adding (1) and (2), we get

$$(494)^2 = (1652 + 1968 \cos\theta_4)^2 + (1968 \sin\theta_4)^2$$

i.e.,

$$(494)^2 = (1652)^2 + 2x1652x1968\cos\theta_4 + (1968 \cos\theta_4)^2 + (1968 \sin\theta_4)^2$$

On solving weget,

$$\cos\theta_4 = -0.978 \quad \text{and} \quad \theta_4 = 167.9^0 \text{ or } 192.1^0$$

Choosing one value, say $\theta_4 = 167.9^0$

Dividing (2) by (1), we get

$$\tan\theta_1 = \frac{1968\sin(167.9^0)}{1652 + 1968\cos(167.9^0)} = \frac{+412.53}{-272.28} = -1.515$$

i.e., $\theta_1 = 123.4^0$

Step 2:

Resolve the forces into their horizontal and vertical components and take their sums. Sum of the horizontal components gives

$$380\ \mathbf{r}\ \mathbf{cos(}123.4^{0}\mathbf{)} + \mathbf{m}_2\ \mathbf{r}\ \mathbf{cos}\ \theta_2 + 590\ \mathbf{rcos}\ 0^{0} + 480\ \mathbf{rcos(}167.9^{0}\mathbf{)} = 0$$

$$\mathbf{or}\ \mathbf{m}_2 \mathbf{cos}\ \theta_2 \quad \mathbf{= 88.5} - - - - - - - - - - - - - - \mathbf{(3)}$$

Sum of the vertical components gives

$$380\ r\sin(123.4^0) + m_2 r \sin\theta_2 + 590\ r\sin 0^0 + 480\ r\sin(167.9^0) = 0$$

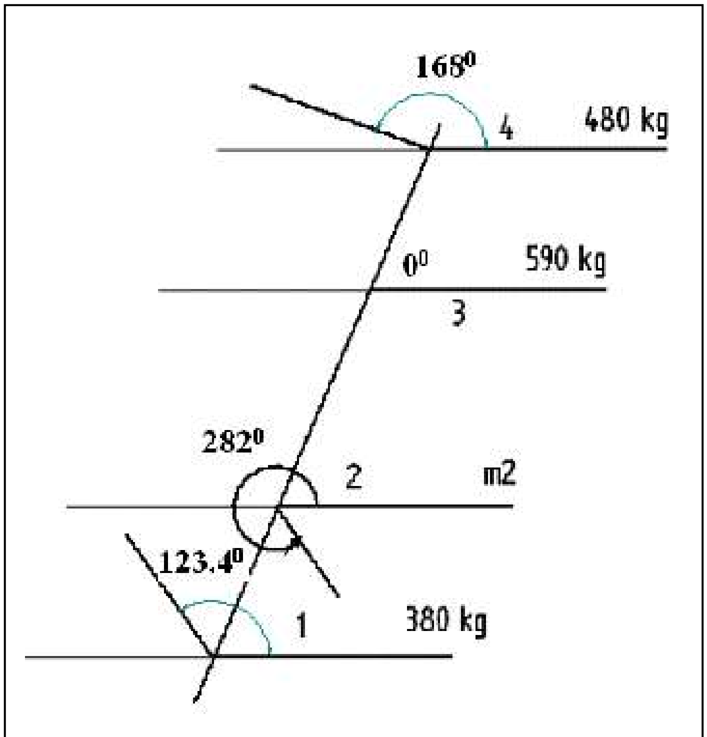
$$\text{or } m_2 \sin\theta_2 = -417.9 \text{ --------------- } (4)$$

Squaring and adding (3) and (4), we get

$$m_2 = 427.1\ \text{kg} \quad \textbf{Ans}$$

Dividing (4) by (3), we get $\tan\theta_2 = \dfrac{-417.9}{88.5} = -4.72$

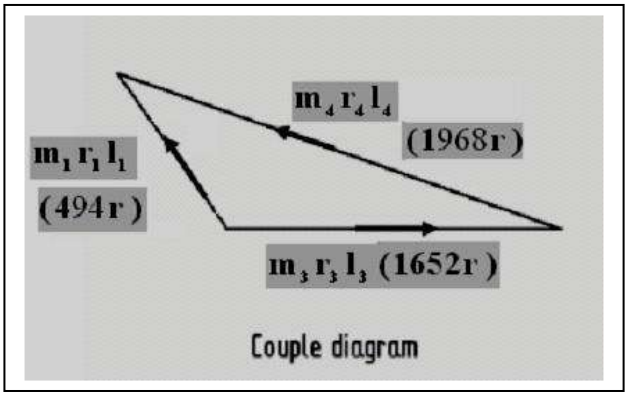
$$\text{or} \quad \theta_2 = 282^0 \quad \textbf{Ans}$$



Relative crank positions

Graphical Method:

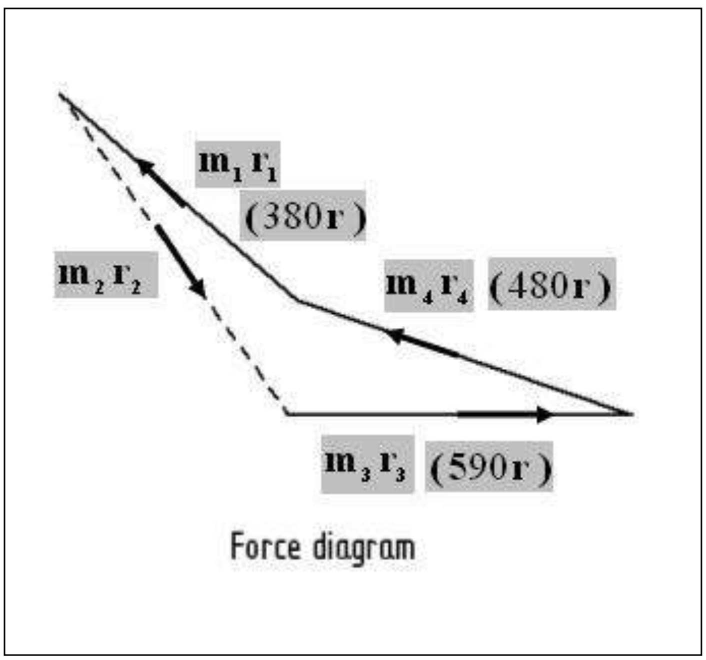
Step 1: Draw the couple diagram taking a suitable scale as shown.



Couple diagram

This diagram provides the relative direction of the masses **m_1, m_3 and m_4**.

Step 2: Now, draw the force polygon taking a suitable scale as shown.



Force diagram

This gives the direction and magnitude of mass m_2.

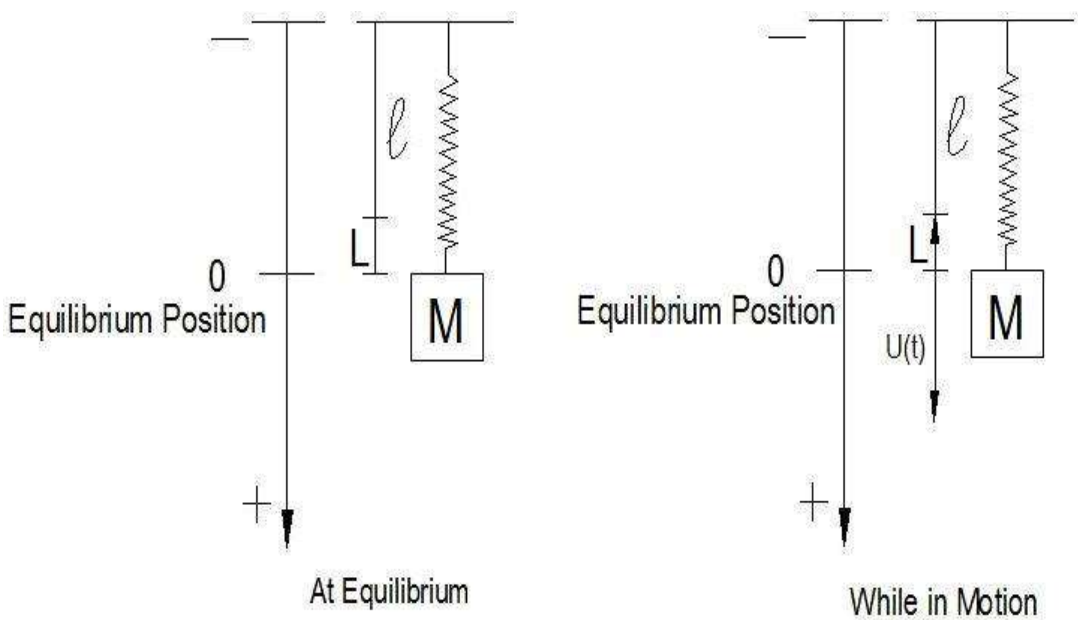
The results are:

$\theta_4 = 168^0$, $\theta_1 = 12$RATIONS3^0, $\theta_2 = 28$

UNIT 5

MECHANICAL VIBRATIONS

A mass m is suspended at the end of a spring, its weight stretches the spring by a length L to reach a static state (the *equilibrium position* of the system). Let $u(t)$ denote the displacement, as a function of time, of the mass relative to its equilibrium position. Recall that the textbook's convention is that downward is positive. Therefore, $u > 0$ means the spring is stretched beyond its equilibrium length, while $u < 0$ means that the spring is compressed. The mass is then assumed to be set in motion (by any one of several means).



The equations that govern a mass-spring system

At equilibrium: (by *Hooke's Law*)

$$mg = kL$$

While in motion:

$$m\, u'' + \gamma\, u' + k\, u = F(t)$$

This is a second order linear differential equation with constant coefficients. It usually comes with two initial conditions: $u(t_0) = u_0$, and $u'(t_0) = u'_0$.

<u>Summary of terms</u>:

$u(t)$ = displacement of the mass relative to its equilibrium position.
m = mass ($m > 0$)

γ = damping constant ($\gamma \geq 0$)

k = spring (Hooke's) constant ($k > 0$)

g = gravitational constant
L = elongation of the spring caused by the weight $F(t)$ = Externally applied forcing function, if any $u(t_0)$ = initial displacement of the mass
$u'(t_0)$ = initial velocity of the mass

Undamped Free Vibration ($\gamma = 0$, $F(t) = 0$)

The simplest mechanical vibration equation occurs when $\gamma = 0$, $F(t) = 0$.
This is the undamped free vibration. The motion equation is

$$m\,u'' + k\,u = 0.$$

The characteristic equation is $mr^2 + k = 0$. Its solutions are The general solution is then $r = \pm\sqrt{\frac{k}{m}}\,i$.

$$u(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.$$

Where $\omega_0 = \sqrt{\frac{k}{m}}$ is called the *natural frequency* of the system. It is the frequency at which the system tends to oscillate in the absence of any damping. A motion of this type is called *simple harmonic motion*. It is a perpetual, sinusoidal, motion.

Comment: Just like everywhere else in calculus, the angle is measured in radians, and the (angular) frequency is given in radians per second. The frequency is **not** given in hertz (which measures the number of cycles or revolutions per second). Instead, their relation is: 2π radians/sec = 1 hertz.

The (natural) *period* of the oscillation is given by $T = \frac{2\pi}{\omega_0}$ (seconds).

To get a clearer picture of how this solution behaves, we can simplify it with trig identities and rewrite it as

$$u(t) = R\cos(\omega_0 t - \delta).$$

The displacement is oscillating steadily with constant amplitude of oscillation

$$R = \sqrt{C_1^2 + C_2^2}.$$

The angle δ is the *phase* or *phase angle* of displacement. It measures how much $u(t)$ lags (when $\delta > 0$), or leads (when $\delta < 0$) relative to $\cos(\omega_0 t)$, which has a peak at $t = 0$. The phase angle satisfies the relation

$$\tan\delta = \frac{C_2}{C_1}.$$

More explicitly, it is calculated by:

$$\delta = \tan^{-1}\frac{C_2}{C_1}, \qquad \text{if } C_1 > 0,$$

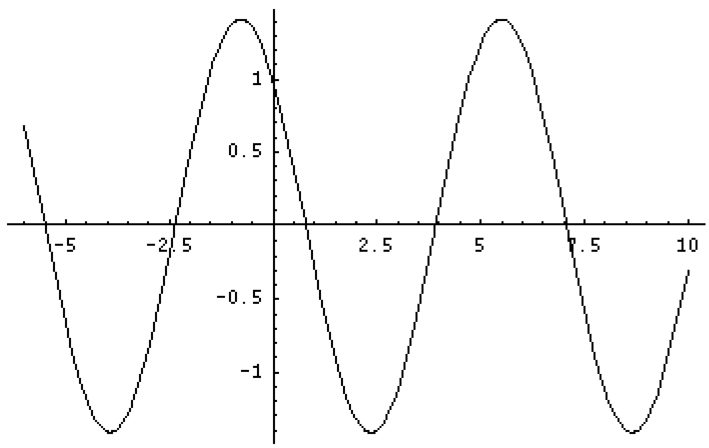
$$\delta = \tan^{-1}\frac{C_2}{C_1} + \pi, \qquad \text{if } C_1 < 0,$$

$$\delta = \frac{\pi}{2}, \qquad \text{if } C_1 = 0 \text{ and } C_2 > 0,$$

$$\delta = -\frac{\pi}{2}, \qquad \text{if } C_1 = 0 \text{ and } C_2 < 0,$$

The angle is undefined if $C_1 = C_2 = 0$.

An example of simple harmonic motion:



Graph of $u(t) = \cos(t) - \sin(t)$

Amplitude: $R = \sqrt{2}$

Phase angle: $\delta = -\pi/4$

Damped Free Vibration ($\gamma > 0$, $F(t) = 0$)

When damping is present (as it realistically always is) the motion equation of the unforced mass-spring system becomes

$$m\,u'' + \gamma\,u' + k\,u = 0.$$

Where m, γ, k are all positive constants. The characteristic equation is $m\,r^2 + \gamma\,r + k = 0$. Its solution(s) will be either negative real numbers, or complex numbers with negative real parts. The displacement $u(t)$ behaves differently depending on the size of γ relative to m and k. There are three possible classes of behaviors based on the possible types of root(s) of the characteristic polynomial.

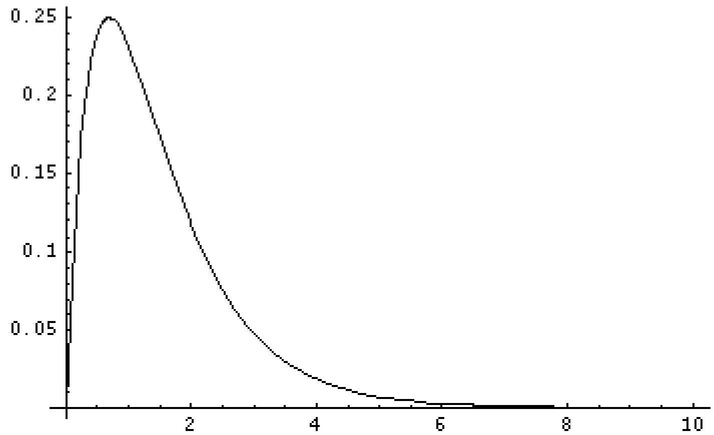
Case I. Two distinct (negative) real roots

When $\gamma^2 > 4mk$, there are two distinct real roots, both are negative. The displacement is in the form

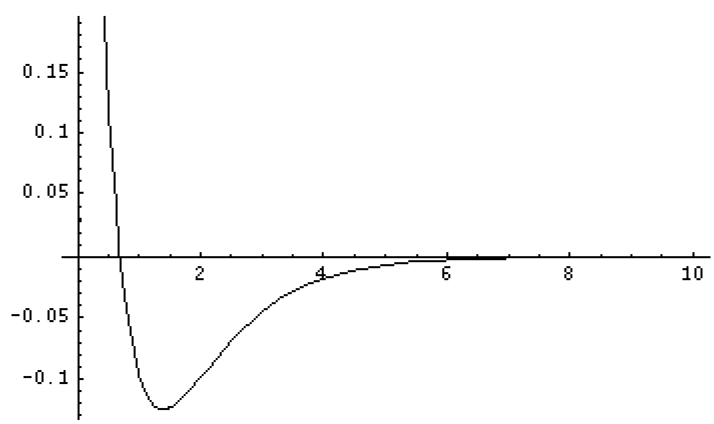
$$u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

A mass-spring system with such type displacement function is called *overdamped*. Note that the system does not oscillate; it has no periodic components in the solution. In fact, depending on the initial conditions the mass of an overdamped mass-spring system might or might not cross over its equilibrium position. But it could cross the equilibrium position at most once.

Figures: Displacement of an Overdamped system



Graph of $u(t) = e^{-t} - e^{-2t}$



Graph of $u(t) = -e^{-t} + 2e^{-2t}$

Case II. One repeated (negative) real root

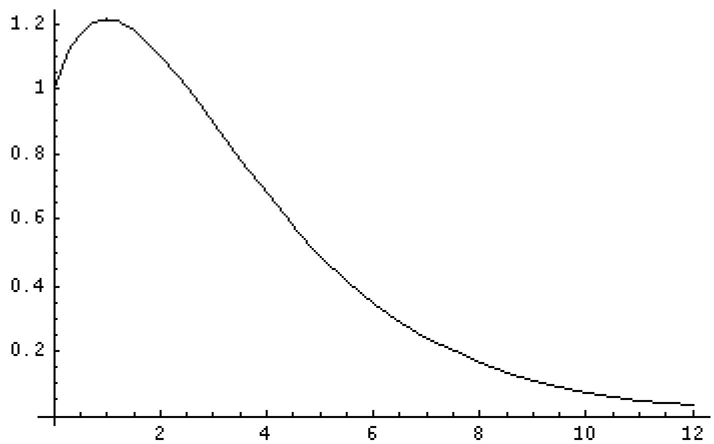
When $\gamma^2 = 4mk$, there is one (repeated) real root. It is negative: $r = \frac{-\gamma}{2m}$. The displacement is in the form

$$u(t) = C_1 e^{rt} + C_2 t e^{rt}.$$

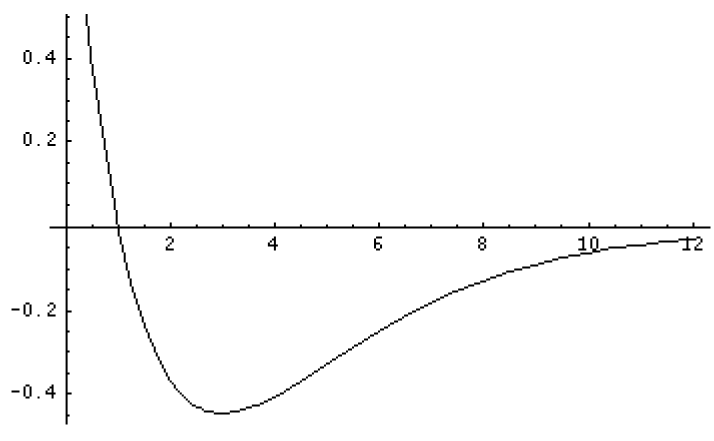
A system exhibits this behavior is called *critically damped*. That is, the damping coefficient γ is just large enough to prevent oscillation. As can be seen, this system does not oscillate, either. Just like the overdamped case, the mass could cross its equilibrium position at most one time.

Comment: The value $\gamma^2 = 4mk \rightarrow \gamma = \sqrt{4mk}$ is called *critical damping*. It is the threshold level below which damping would be too small to prevent the system from oscillating.

Figures: Displacement of a Critically Damped system



Graph of $u(t) = e^{-t/2} + t\,e^{-t/2}$



Graph of $u(t) = e^{-t/2} - t\,e^{-t/2}$

Case III. Two complex conjugate roots

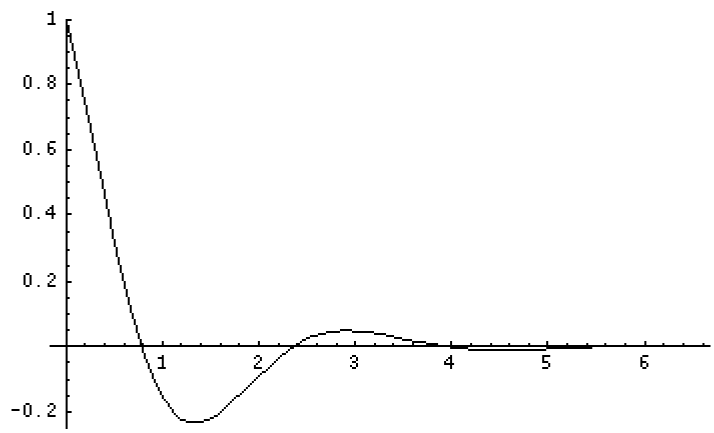
When $\gamma^2 < 4mk$, there are two complex conjugate roots, where their common real part, λ, is always negative. The displacement is in the form

$$u(t) = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t.$$

A system exhibits this behavior is called *underdamped*. The name means that the damping is small compares to *m* and *k*, and as a result vibrations will occur. The system oscillates (note the sinusoidal components in the solution). The displacement function can be rewritten as

$$u(t) = R\, e^{\lambda\, t} \cos\,(\mu t - \delta).$$

The formulas for R and δ are the same as in the previous (undamped free vibration) section. The displacement function is oscillating, but the amplitude of oscillation, $Re^{\lambda t}$, is decaying exponentially. For all particular solutions (except the zero solution that corresponds to the initial conditions $u(t_0) = 0$, $u'(t_0) = 0$), the mass crosses its equilibrium position infinitely often.



Damped oscillation: $u(t) = e^{-t} \cos(2t)$

The displacement of an underdamped mass-spring system is a *quasi-periodic* function (that is, it shows periodic-like motion, but it is not truly periodic because its amplitude is ever decreasing so it does not exactly repeat itself). It is oscillating at *quasi-frequency*, which is μ radians per second. (It's just the frequency of the sinusoidal components of the displacement.) The peak-to-peak time of the oscillation is the *quasi-period*: (seconds).

$$T_q = \frac{2\pi}{\mu}$$

In addition to cause the amplitude to gradually decay to zero, damping has another, more subtle, effect on the oscillating motion: It immediately decreases the quasi-frequency and, therefore, lengthens the quasi-period (compare to the natural frequency and natural period of an undamped system). The larger the damping constant γ, the smaller quasi-frequency and the longer the quasi-period become. Eventually, at the critical damping threshold, when $\gamma = \sqrt{4mk}$, the quasi-frequency vanishes and the displacement becomes aperiodic (becoming instead a critically damped system).

Note that <u>in all 3 cases of damped free vibration, the displacement function</u> <u>tends to zero as $t \to \infty$</u>. This behavior makes perfect sense from a conservation of energy point-of-view: while the system is in motion, the damping wastes away whatever energy the system has started out with, but there is no forcing function to supply the system with additional energy.
Consequently, eventually the motion comes to a halt.

Example: A mass of 1 kg stretches a spring 0.1 m. The system has a damping constant of $\gamma = 14$. At $t = 0$, the mass is pulled down 2 m and released with an upward velocity of 3.5 m/s. Find the displacement function. What are the system's quasi-frequency and quasi-period?

$m = 1$, $\gamma = 14$, $L = 0.1$;

$mg = 9.8 = kL = 0.1\,k \quad \rightarrow \quad 98 = k.$

The motion equation is $u'' + 14u' + 98u = 0$, and the initial conditions are $u(0) = 2$, $u'(0) = -3.5$.

The roots of characteristic polynomial are $r = -7 \pm 7i$:

$$u(t) = C_1 e^{-7t} \cos 7t + C_2 e^{-7t} \sin 7t$$

Therefore, the quasi-frequency is 7 (rad/sec) and the quasi-period is

$$T_q = \frac{2\pi}{7} \quad \text{(seconds)}.$$

Apply the initial condition and we get $C_1 = 2$, and $C_2 = 3/2$. Hence

$$u(t) = 2e^{-7t} \cos 7t + 1.5e^{-7t} \sin 7t.$$

Summary: the Effects of Damping on an Unforced Mass-Spring System

Consider a mass-spring system undergoing free vibration (i.e. without a forcing function) described by the equation:

$$m u'' + \gamma u' + k u = 0, \qquad m > 0, \qquad k > 0.$$

The behavior of the system is determined by the magnitude of the damping coefficient γ relative to m and k.

1. Undamped system (when $\gamma = 0$)

   Displacement: $u(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$

   Oscillation: Yes, periodic (at natural frequency $\omega_0 = \sqrt{\frac{k}{m}}$)

   Notes: Steady oscillation with constant amplitude $R = \sqrt{C_1^2 + C_2^2}$.

2. Underdamped system (when $0 < \gamma^2 < 4mk$)

   Displacement: $u(t) = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$

   Oscillation: Yes, quasi-periodic (at quasi-frequency μ)
   Notes: Exponentially-decaying oscillation

3. Critically Damped system (when $\gamma^2 = 4mk$)

   Displacement: $u(t) = C_1 e^{rt} + C_2 t e^{rt}$

   Oscillation: No

4. Overdamped system (when $\gamma^2 > 4mk$)

   Displacement: $u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

   Oscillation: No

Mechanical Vibrations, $F(t) = 0$

γ

$\gamma^2 > 4mk$

Overdamped

No Oscillation,

Displacement: $u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$,
Mass crosses equilibrium at most once.

$\gamma^2 = 4mk$ **Critically Damped**

No Oscillation

Displacement: $u(t) = C_1 e^{rt} + C_2 te^{rt}$

Mass crosses equilibrium at most once.

$\gamma^2 < 4mk$

Underdamped

System oscillates with amplitude decreasing exponentially overtime,

Displacement: $u(t) = C_1 e^{\lambda t} \cos \mu t + C_2 e^{\lambda t} \sin \mu t$,

Oscillation quasi periodic: $T_q = 2\pi/\mu$

DAMPING INCREASES

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γ=0

Undamped

γ = 0, **Displacement:** $u(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$

Natural frequency: $\omega_0 =$, **Steady oscillation** with constant amplitude

Forced Vibrations

Undamped Forced Vibration ($\gamma = 0$, $F(t) \neq 0$)

Now let us introduce a nonzero forcing function into the mass-spring system.
To keep things simple, let damping coefficient $\gamma = 0$. The motion equation is

$$m\,u'' + k\,u = F(t).$$

In particular, we are most interested in the cases where $F(t)$ is a periodic function. Without the losses of generality, let us assume that the forcing function is some multiple of cosine:

$$m\,u'' + k\,u = F_0 \cos \omega t.$$

This is a nonhomogeneous linear equation with the complementary solution

$$u_c(t) = C_1 \cos \omega_0\, t + C_2 \sin \omega_0\, t.$$

The form of the particular solution that the displacement function will have depends on the value of the forcing function's frequency, ω.

Case I. When $\omega \neq \omega_0$

If $\omega \neq \omega_0$ then the form of the particular solution corresponding to the forcing function is

$$Y = A \cos \omega t + B \sin \omega t.$$

Solving for A and B using the method of Undetermined Coefficients, we find that

$$Y = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

Therefore, the general solution of the displacement function is

$$u(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

An interesting instance of such a forced vibration occurs when the initial conditions are $u(0) = 0$, and $u'(0) = 0$. Applying the initial conditions to the general solution and we get

$$C_1 = \frac{-F_0}{m(\omega_0^2 - \omega^2)}, \qquad \text{and} \qquad C_2 = 0.$$

Thus,

$$u(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)}\left(\cos\omega t - \cos\omega_0 t\right).$$

Again, a clearer picture of the behavior of this solution can be obtained by rewriting it, using the identity:

$$\sin(A)\sin(B) = [\cos(A - B) - \cos(A + B)] / 2.$$

The displacement becomes

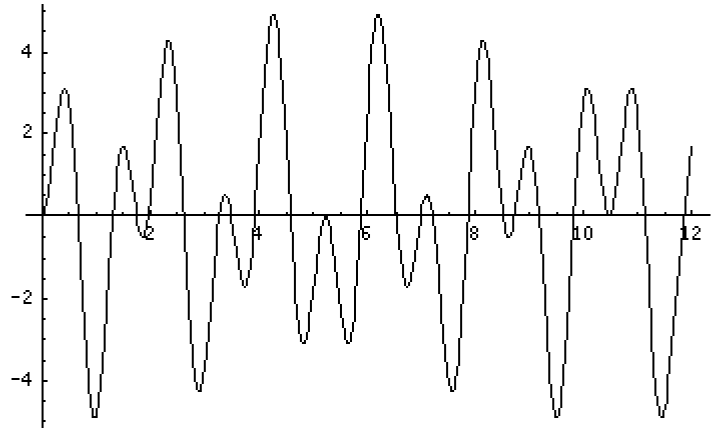
$$u(t) = \left[\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\frac{(\omega_0 - \omega)t}{2}\right] \sin\frac{(\omega_0 + \omega)t}{2}.$$

The behavior exhibited by this function is that the higher-frequency, of $(\omega_0 + \omega) / 2$, sine curve sees its amplitude of oscillation modified by its lower-frequency, of $(\omega_0 - \omega)/2$, counterpart.

This type of behavior, where an oscillating motion's own amplitude shows periodic variation, is called a *beat*. The quantity $\omega_b = | \omega_0 - \omega |$ is called the *beat frequency*. It can be derived by dividing 2π by the distance between

adjacent zeros of $\sin\dfrac{(\omega_0 - \omega)t}{2}$.

An example of beat:



Graph of $u(t) = 5 \sin(1.8t) \sin(4.8t)$

Case II. When $\omega = \omega_0$

If the periodic forcing function has the same frequency as the natural frequency, that is $\omega = \omega_0$, then the form of the particular solution becomes

$$Y = A\, t \cos \omega_0 t + B\, t \sin \omega_0 t.$$

Use the method of Undetermined Coefficients we can find that

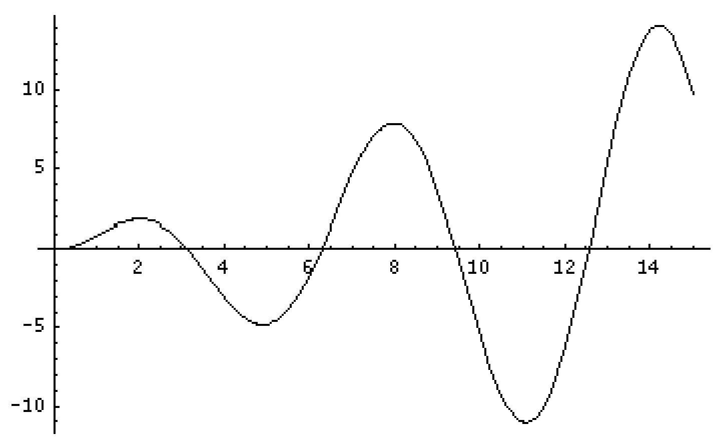
$$A = 0, \qquad \text{and} \qquad B = \frac{F_0}{2m\omega_0}.$$

The general solution is, therefore,

$$u(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{F_0}{2m\omega_0} t \sin \omega_0 t$$

.

The first two terms in the solution, as seen previously, could be combined to become a cosine term $u(t) = R \cos(\omega_0 t - \delta)$, of steady oscillation. The third term, however, is a sinusoidal wave whose amplitude increases proportionally with elapsed time. This phenomenon is called *resonance*.



Resonance: graph of $u(t) = t\sin(t)$

Technically, true resonance only occurs if <u>all</u> of the conditions below are satisfied:

1. There is no damping: $\gamma = 0$,
2. A periodic forcing function is present, and
3. The frequency of the forcing function exactly matches the natural frequency of the mass-spring system.

However, similar behaviors, of unexpectedly large amplitude of oscillation due to a fairly low-strength forcing function occur when damping is present but is very small, and/or when the frequency of forcing function is very close to the natural frequency of the system.